

# The Periodic Joint Replenishment Problem is Strongly $\mathcal{NP}$ -Hard

Cohen Tamar<sup>1</sup>, Yedidsion Liron<sup>2\*</sup>

<sup>1</sup>Operation Research Center,  
MIT- Massachusetts Institute of Technology,  
MA, USA.

<sup>2</sup>Faculty of Industrial Engineering and Management,  
Technion - Israel Institute of Technology,  
Haifa, Israel.

## Abstract

In this paper we study the long-standing open question regarding the computational complexity of one of the core problems in supply chains management, the periodic joint replenishment problem. This problem has received a lot of attention over the years and many heuristic and approximation algorithms were suggested. However, in spite of the vast effort, the complexity of the problem remained unresolved. In this paper, we provide a proof that the problem is indeed strongly  $\mathcal{NP}$ -hard.

## 1 Introduction

Many inventory models are aimed at minimizing ordering and holding costs while satisfying demand. The Joint Replenishment Problem (JRP) deals with the prospect of saving resources through coordinated replenishments in order to achieve substantial cost savings. In this research we study the complexity of JRP. In the JRP one is required to schedule the replenishment times of numerous commodities (sometimes called items or products) in order to supply an external demand per commodity. We refer to the schedule of the replenishment times as the ordering policy. Each commodity incurs fixed ordering costs every time it is replenished as well as linear holding costs that are proportional to the quantity of the commodity held in storage. Linking all commodities, a joint ordering cost is incurred whenever one or more commodities are ordered. The objective of JRP is to minimize the sum of ordering and holding costs. It is a natural extension of the classical economic lot-sizing model that considers the optimal trade-off between ordering costs and holding costs for a single commodity. With multiple commodities, JRP adds the possibility of saving resources via coordinated replenishments, a common phenomenon in supply chain management. JRP is a special case of One-Warehouse-N-Retailers problem (OWNR), which deals with a single warehouse receiving goods from an external supplier and distributing to multiple retailers. The warehouse could also serve as a storage point. JRP in particular is a special case of the OWNRR with a very high warehouse holding cost.

There are some distinctions between variations of JRP.

- Commodity order policy constraints: There are 3 types of order policy constraints for the JRP. The first model requires a periodic ordering policy. A periodic ordering policy is one in which for each commodity we must determine a cycle time. An order will occur at each multiple of that cycle time. We refer to this model as the periodic JRP (PJRP). The second model does not require a cycle time for each commodity; however it requires a cyclic ordering policy. We refer to this model as the cyclic JRP (CJRP). The last model has no limits on the ordering policy. Note that PJRP is a constrained version of CJRP, which in turn is a constrained version of the ordering policy JRP. In this research we focus on PJRP.

---

\*Corresponding author. Email - lirony@ie.technion.ac.il

- **Joint order policy constraints:** The joint ordering cost in the PJRP model is a complicated function of the inter-replenishment times, so it is often assumed that joint orders are placed periodically, even if some joint orders are empty, and that the cycle times of the commodities are always a multiple of the joint order cycle time. We denote this type of policy as the General Integer model (GI), while policies with no joint order constraints are referred to as the General Integer model with Correction Factor (GICF). The PJRP with GI policy constraint is also referred to as Strict PJRP. We refer to PJRP with GICF policy constraint as the General PJRP or simply as PJRP. Note that Strict PJRP is a constrained version of General PJRP. In this research we focus on the General PJRP.
- **Demand type:** Another important distinction is between problems with stationary demand for each commodity and problems with fluctuating demand. Note that the problem with stationary demand is a special case of the problem with fluctuating demand. In this research we focus on the set of problems with stationary demand.
- **Time horizon:** The time horizon defines the horizon for which one must plan an order policy. We distinguish between the problem with infinite horizon and the problem with finite horizon. Most of the research over the years focused on JRP with an infinite horizon, specifically when considering stationary demand. The motivation for considering a finite horizon with non-stationary demand comes from knowing the demand only for a finite horizon. This motivation does not apply for stationary demand.
- **Solution integrality:** The integrality of the solution determines whether the ordering policy will be integral or not. Note that the integral problem is a constrained version of the continuance problem. In this research we focus on the integral problem.

## 1.1 Literature review

As far as we know, the complexity of JRP with stationary demands remained open for all models. Some papers addressing JRP with constant demands (e.g., [17], [22]) mistakenly cite a result by Arkin et al. [1], which proves that JRP with non-stationary demands is  $\mathcal{NP}$ -hard. Arkin et al. [1] stated that since JRP is a special case of OWNRR, proving JRP hardness also proves the hardness of OWNRR. Lately, Schulz and Telha [38] have proved that finding an optimal replenishment policy for the stationary PJRP is at least as hard as the integer factorization problem. When referring to the existence of a polynomial-time optimal algorithm for the stationary PJRP, Schulz and Telha also stated that this case remains open. In this paper we show that the PJRP with stationary demands is strongly  $\mathcal{NP}$ -hard.

**Strict PJRP.** The problem of Strict PJRP was well covered in the reviews by Goyal and Satir [7] and Khouja and Goyal [16]. Many research attempts have been made to find efficient solutions to Strict PJRP. In the early 1970s, two pioneer studies suggested a graphical heuristic approach [41], [26]. At the same time, Goyal [3] had suggested a non-polynomial lower bound based heuristics to find the optimal strict cyclic policy, in which the cycle time for each commodity is the joint replenishment cycle time. Van Eijs [44] suggested a modified version of Goyal’s algorithm that involved using a non-strict cyclic policy.

Based on these studies, many heuristics have been developed to solve the Strict PJRP. Silver [42] developed a heuristic algorithm to find the joint period cycle time. Following this algorithm many iterative search heuristics were suggested with different search bounds (Kaspi and Rosenblatt [13], Goyal and Belton [6], Kaspi and Rosenblatt [14], Goyal and Deshmukh [5], Viswanathan [45], Fung and Ma [2] that was later modified by [46], [32]). Wildeman et al. [47] used the idea of the iterative search and implemented it in a heuristic that converges to an optimal solution. For certain values of the joint period cycle time they solved Strict PJRP optimally using a Lipschitz optimization procedure. Another heuristic approach for the problem was developed by Olsen [29], called evolutionary algorithm.

Since JRP is a special case of OWNRR, results regarding the OWNRR hold for JRP as well. Hence, the following results are applicable for JRP. A prominent advancement in the study of OWNRR, the optimal Power-of-Two policy, was achieved by Roundy [37]. This policy could be computed in  $O(n \log n)$  time. Roundy proved that the cost of the best power-of-two policy can achieve 98% of an optimal policy (94% if the base planning period is fixed). In other words, he suggested a 1.02-approximation (1.064 for the fixed based planning period) for JRP, where a  $\rho$ -approximation algorithm is an algorithm that is polynomial with

respect to the number of elements, and the ratio between the worst case scenario solution and the optimal solution is bounded by a constant,  $\rho$ . Note that fixed based planning period implies an integral model, while the general power-of-two policy allows a non-integral solution. Based on Roundy’s findings, Jackson et al. [11] proposed an efficient algorithm that offers a replenishment policy in which the cost is within a factor of  $\sqrt{\frac{9}{8}} \approx 1.06$  of the optimal solution. This approximation was later improved to  $\frac{1}{\sqrt{2} \log 2}$  for a non-fixed based planning period [24].

Several studies have been made based on the Power-of-Two policy, including Lee and Yao [17], Muckstadt and Roundy [23], Teo and Bertsimas [43]. Teo and Bertsimas have also noted in their paper that finding the optimal lot sizing policies for stationary demand lot sizing problems is still an open issue.

Lu and Posner [20] presented a fully polynomial time approximation scheme (FPTAS) for the Strict PJRP model with fixed base. Later, Segev [39] presented a quasi-polynomial-time approximation scheme (QPTAS), which shows that the problem is most likely not  $\mathcal{APX}$ -hard. In addition, efficient polynomial time approximation scheme (EPTAS) for JRP with finite time horizon and stationary demand was presented by Nonner and Sviridenko [28].

This problem was researched in many other different setups, such as JRP under resource constraints (Goyal [4], Khouja et al [15] and Moon and Cha [22]), minimum order quantities (Porras and Dekker [34]) and non-stationary holding cost (Levi et al. [19], Nonner and Souza [27], Levi et al. [18]).

**General PJRP.** Porras and Dekker [33] pointed out that adding the correction factor leads to a completely different problem, at least in terms of exact solvability. Porras and Dekker [32] show that changing the model from Strict PJRP to PJRP significantly changes the joint replenishment cycles and the commodities replenishment cycles. The difference in solvability is evidenced by the sheer number of decision variables. In the Strict PJRP all commodities cycle times are simple functions of the joint replenishment cycle time. Thus there is actually only a single decision variable. However, this is not the case with the PJRP where we have  $n$  decision variables, one for each commodity. We believe this to be the main reason for the difference in the amount of research conducted on Strict PJRP with respect to the PJRP despite the PJRP being more practical. In practice, Strict PJRP is much less common than PJRP as it involves paying for empty deliveries. Strict PJRP may occur only if there is a binding contract with a delivery company. Although such a binding contract may decrease the cost of the joint replenishment significantly, it usually limits the flexibility of choosing the joint replenishment cycles. Lately, Schulz and Telha [38] presented a polynomial time approximation scheme (PTAS) for the PJRP case.

**Finite horizon.** Several heuristics were designed to deal with the finite horizon model. Most of the finite time heuristics assume variable demands and run-in time  $\Omega(T)$  [19],[12]. Schulz and Telha [38] presented a polynomial-time  $\sqrt{9/8}$ -approximation algorithm for the JRP with dynamic policies and finite horizon. As the time horizon  $T$  increases, the ratio converges to  $\sqrt{9/8}$ . Schulz and Telha [38] also presented an FPTAS for the Strict PJRP case with no fixed base and a finite time horizon.

Our paper proceeds as follows: In Section 2 we formulate the problem. In Section 3 we prove that the infinite horizon PJRP is strongly  $\mathcal{NP}$ -hard. In Section 4 we show why the finite horizon PJRP is  $\mathcal{NP}$ -hard (not necessarily in the strong sense). Section 5 Summarizes the paper and discusses other related open problems.

## 2 Model Formulation

In this research, we consider the case of an infinite time horizon, and a system composed of several commodities, for each of which there is an external stationary demand. The demand has to be satisfied in each period. Backlogging and lost sales are not allowed. Each commodity incurs a fixed ordering cost for each period in which an order of the commodity is placed, as well as a linear inventory holding cost for each period a unit of commodity remains in storage. In addition, a joint ordering cost is incurred for each time period where one or more orders are placed. We use the following notations, where the units are given in

square brackets:

$$\begin{aligned}
N &- \text{Number of commodities in the system } [units], \\
\lambda_c &- \text{Demand rate for commodity } c \text{ per period. } \left[ \frac{units}{period} \right], \\
h_c &- \text{Holding cost for commodity } c \text{ per period. } \left[ \frac{\$}{units \cdot period} \right], \\
K_c &- \text{Fixed ordering cost for commodity } c \text{ } [\$], \\
q_c &- \text{Order quantity for commodity } c \text{ } [units], \\
K_0 &- \text{Fixed joint ordering cost } [\$].
\end{aligned}$$

The objective is to find an integer ordering cycle time,  $t_c$ , for each commodity  $c$  so as to minimize the periodic sum of ordering and holding costs of all commodities.

The simple model, in which there is only 1 commodity, is known as the Economic Order Quantity (EOQ). While examining commodity  $c$ , we define its *standalone problem* as the optimal ordering quantity problem for a single commodity  $c$  with no joint setup cost and an infinite horizon. The standalone problem is a simple EOQ problem.

The EOQ model assumes without loss of generality that the on hand inventory at time zero is zero. Shortage is not allowed, so we must place an order at time zero. The average periodic cost, as a function of the cycle time  $t_c$ , denoted by  $g(t_c)$  is given by

$$g(t_c) = \frac{K_c}{t_c} + \lambda_c h_c \frac{t_c}{2}, \quad (1)$$

and the optimal cycle time for  $g(t_c)$ , denoted by  $t_c^*$  is

$$t_c^* = \sqrt{\frac{2K_c}{h_c \lambda_c}}. \quad (2)$$

In addition,  $g(t_c)$  where  $t_c = \beta t_c^*$  (for an arbitrary constant  $\beta$ ) could also be calculated using:

$$g(\beta t_c^*) = \frac{1}{2} \left( \frac{1}{\beta} + \beta \right) g(t_c^*).$$

Accordingly, when debating between two options for a cycle time  $\underline{t}_c$  and  $\bar{t}_c$ , such that  $\underline{t}_c < t_c^* < \bar{t}_c$ , then based on the standalone total cost our choice would be:

$$\begin{aligned}
&\bar{t}_c && \text{if } \sqrt{\underline{t}_c \bar{t}_c} > t_c^* \\
&\underline{t}_c && \text{if } \sqrt{\underline{t}_c \bar{t}_c} \leq t_c^*.
\end{aligned} \quad (3)$$

Without loss of generality, throughout this research, we assume that  $\lambda_c = 2$  for all commodities.

See full elaboration and additional analysis in [25] and [49].

### 3 $\mathcal{NP}$ -Hardness of the PJRP.

#### 3.1 A reduction from 3SAT to PJRP

In this section we present a reduction from 3SAT to the PJRP with infinite horizon. The 3SAT is defined as follows:

**Definition 1** *Given a logical expression,  $\varphi$ , in a CNF form with  $m$  clauses and  $n$  variables,  $x_1, \dots, x_n$ , where each clause,  $(z_i \cup z_j \cup z_s)$  where  $z_i \in \{x_i, \bar{x}_i\}$ , contains exactly 3 literals, is there a feasible assignment to the variables such that each clause contains at least one true literal?*

The 3SAT is strongly  $\mathcal{NP}$ -hard [30].

In this reduction we use pairs of prime numbers with a difference of at most  $b$  between them where  $b$  is a constant. To even consider such a reduction we have to make sure that such a set exists for any input size  $n$  and that it can be found in polynomial time. To do so we use the breakthrough proof by Zhang [48]. Zhang proved that there is an infinite number of 2-tuple( $b$ ), prime pairs with  $b \leq 7 \cdot 10^7$ , where  $k$ -tuple( $J$ ) primes are a finite collection of  $k = |J|$  values representing a repeatable pattern  $J$  of differences between prime numbers. In addition, Zhang's proof could be used to show that  $b$  is associated with another constant  $\tilde{b}$  such that there are at least

$$\frac{x}{\log^{\tilde{b}} x}$$

2-tuple( $b$ ) prime pairs smaller than  $x$ . Since Zhang's proof, attempts were made to decrease the bounds on both  $b$  and  $\tilde{b}$ . The latest result, attained by the *Polymath8* project [31], sets the bounds  $b \leq 256$  and  $\tilde{b} \leq 50$ .

An important special case of 2-tuple( $b$ ) where  $b = 2$  is twin primes. That is, twin primes are pairs of consecutive prime numbers with a difference of exactly 2 between them. The twin prime conjecture [8] and the first Hardy-Littlewood conjecture [9] maintain that  $b = 2$  and  $\tilde{b} = 2$ . These values would surely make our proof simpler. However, for sake of comprehensiveness we use general constants  $b$  and  $\tilde{b}$ .

In our proof we require a set of  $n$  pairs of primes, denoted by  $(\underline{p}_1, \bar{p}_1), (\underline{p}_2, \bar{p}_2), \dots, (\underline{p}_n, \bar{p}_n)$  such that  $\underline{p}_1 < \bar{p}_1 < \dots < \underline{p}_n < \bar{p}_n$ . We denote the set of primes  $\{\underline{p}_1, \bar{p}_1, \dots, \underline{p}_n, \bar{p}_n\}$  by  $VP$  and the set of pairs  $\{(\underline{p}_1, \bar{p}_1), \dots, (\underline{p}_n, \bar{p}_n)\}$  by  $VP_2$ . The primes of  $VP$  and  $VP_2$  have to satisfy the following conditions:

**Condition 1** *The difference between the elements of a pair of consecutive primes, denoted  $b_i = \bar{p}_i - \underline{p}_i$  for  $j = 1, \dots, n$ , is not greater than  $b$ .*

**Condition 2**  $\bar{p}_n < B \cdot \underline{p}_1$  where  $B \geq (6\tilde{b} \log n)^{\tilde{b}}$ .

**Condition 3**  $\underline{p}_1 > n^{6\tilde{b}}$ .

**Condition 4** *Any multiplication of some prime  $p \in VP$  does not fall in-between any pair  $(\underline{p}_i, \bar{p}_i) \in VP_2$ . That is,*

$$\nexists p \in VP, (\underline{p}_i, \bar{p}_i) \in VP_2, \xi \in \mathbb{N} : \underline{p}_i < \xi \cdot p < \bar{p}_i.$$

**Lemma 1** *The set  $VP$  that satisfies Conditions 1-4 could be found in  $O(n^{6\tilde{b}+1} \log^{\tilde{b}} n)$  time.<sup>1</sup>*

Given an input of 3SAT problem, denoted by  $\varphi$ , with  $n$  variables. We find a set of  $n$  pairs of prime numbers that satisfy Conditions 1-4. We associate each pair with a variable of  $\varphi$ . The function  $P(\cdot)$  is defined for each of the variables and their negations in the CNF expression  $\varphi$  as follows:

$$\begin{aligned} P(\bar{x}_i) &= \underline{p}_i \\ P(x_i) &= \bar{p}_i \end{aligned}$$

where  $(\underline{p}_i, \bar{p}_i)$  is the  $i$ -th prime pair in  $VP_2$ . We also define the set  $PP$  of all the prime numbers that are smaller than  $\underline{p}_1$ . In other words,  $PP = \{p : p < \underline{p}_1, p \text{ is prime}\}$ . Let us segment the time horizon into 3 intervals as showed in Figure 1. The first segment, denoted by  $P$ , covers the interval  $P = [0, \underline{p}_1)$ . The second segment, denoted by  $V$ , covers the interval  $V = [\underline{p}_1, \bar{p}_n]$ . The last segment, denoted by  $R$ , covers the interval  $R = (\bar{p}_n, \infty)$ . Note that  $PP \in P$  and  $VP \in V$ .

---

<sup>1</sup>See proof in the Appendix.

Figure 1: Time horizon segmentation

For convenience reasons we define the following quantities:

$$\alpha_c = \prod_{p \in PP} \left( \frac{p-1}{p} \right) \quad (4)$$

$$\alpha_v = \prod_{p \in VP} \left( \frac{p-1}{p} \right) \quad (5)$$

$$\bar{\alpha}_v = \prod_{c_i^x \in Variable} \left( \frac{\bar{p}_i - 1}{\bar{p}_i} \right) \quad (6)$$

$$\underline{\alpha}_v = \prod_{c_i^x \in Variable} \left( \frac{\underline{p}_i - 1}{\underline{p}_i} \right) \quad (7)$$

$$a_n = \prod_{\substack{p_{[j]} \in PP \\ j < n}} \left( 1 - \frac{1}{p_{[j]}} \right), \quad (8)$$

where  $p_{[j]}$  is the  $j^{th}$  largest prime number.

According to the time horizon segmentation we define the *PJRP* instance, denoted by  $\Gamma$  with 3 sets of commodities:

- The first set, denoted by *Constants*, contains commodities with costs constructed such that their optimal cycle time is identical to their standalone optimal cycle time, regardless of the cycle time of the other commodities. The set *Constants* contains commodities of the form  $c_{lm}^{pv}$  for each combination of  $p_l \in PP$  and  $v_m \in VP$ . The standalone optimal cycle time for a commodity  $c_{lm}^{pv}$  is

$$t_{c_{lm}^{pv}}^* = p_l \cdot v_m.$$

The holding cost ( $h_{c_{lm}^{pv}}$ ) and ordering cost ( $K_{c_{lm}^{pv}}$ ) for each commodity  $c_{lm}^{pv} \in Constants$  are as follows:

$$h_{c_{lm}^{pv}} = 1 \quad (9)$$

$$K_{c_{lm}^{pv}} = \left( t_{c_{lm}^{pv}}^* \right)^2 - \frac{1}{2}. \quad (10)$$

- The second set, denoted by *Variables*, contains a commodity  $c_i^x$  for each variable  $x_i$ . We set the costs so that in any optimal solution the cycle time of each commodity corresponding to variable  $x_i$  is either  $\underline{p}_i$  or  $\bar{p}_i$ .

The holding cost ( $h_{c_i^x}$ ) and ordering cost ( $K_{c_i^x}$ ) for each commodity  $c_i^x \in Variables$  are as follows:

$$h_{c_i^x} = \alpha_c \frac{\underline{p}_i^2 - b_i^2}{\underline{p}_i \left( \underline{p}_i + \frac{b_i}{2} \right) \frac{b_i}{2}} \quad (11)$$

$$K_{c_i^x} = h_{c_i^x} \cdot \underline{p}_i \left( \underline{p}_i + b_i \right) - \frac{\underline{p}_i + b_i}{\underline{p}_i + b_i - 1} \alpha_c \bar{\alpha}_v. \quad (12)$$

- The third set, denoted by *Clauses*, contains a commodity  $c_r^\omega$  for each clause  $\omega_r = (z_i \cup z_j \cup z_s)$ . The standalone optimal cycle time for a commodity  $c_r^\omega$  is

$$t_{c_r^\omega}^* = P(z_i) \cdot P(z_j) \cdot P(z_s). \quad (13)$$

The holding cost ( $h_{c_r^\omega}$ ) and ordering cost ( $K_{c_r^\omega}$ ) for each commodity  $c_r^\omega \in \text{Clauses}$  are as follows:

$$h_{c_r^\omega} = 1 \quad (14)$$

$$K_{c_r^\omega} = \left(t_{c_r^\omega}^*\right)^2 - \frac{1}{2}. \quad (15)$$

We set the joint ordering cost to be:

$$K_0 = 1. \quad (16)$$

### 3.2 Optimality analysis

In this section we analyze the characteristics of the optimal solution to  $\Gamma$ . Throughout the remainder of the manuscript we use sensitivity analysis to determine the optimality of certain cycle times. Due to the convex nature of the cost function in Eq. (1) and the discrete nature of our model, in many of our proofs it is sufficient to use sensitivity analysis on cycle times that are within  $\pm 1$  of the optimal standalone solution. To simplify our analysis, we define the function  $\Delta_c(t_c, S)$  that describes the marginal average periodic cost associated with commodity  $c$ 's cycle time,  $t_c$ , and a solution  $S$  to the other commodities in the system. We denote the lower and upper bounds on  $\Delta_c(t_c, S)$  as  $LB(\Delta_c(t_c, S))$  and  $UB(\Delta_c(t_c, S))$ , respectively. We also define  $LB(\Delta_c(t_c))$  and  $UB(\Delta_c(t_c))$  as the lower and upper bounds on the marginal average periodic cost associated with any solution  $S$  to the other commodities in the system and with commodity  $c$ 's cycle time,  $t_c$ .

In the next subsections we prove that solving  $\Gamma$  optimally is equivalent to solving  $\varphi$ . In Section 3.2.1 we show that the cycle times of the commodities in *Constants* and *Clauses* are independent of the cycle times of any other commodity in the problem. In Section 3.2.2 we show that in any optimal solution, the cycle time of each commodity  $c_i^x \in \text{Variables}$  is either  $\underline{p}_i$  or  $\overline{p}_i$ . A selection of a cycle time  $\underline{p}_i$  or  $\overline{p}_i$  for commodity  $c_i^x \in \text{Variables}$  is associated with assigning variable  $x_i$  to either be *false* or *true* in  $\varphi$ , respectively. In Section 3.2.3 we finalize the proof that solving  $\Gamma$  optimally is equivalent to solving  $\varphi$  by showing that in an optimal solution the cycle times of the commodities in *Variables* defines a solution to  $\varphi$  if there is one.

#### 3.2.1 Cycle time of commodities of types *Constants* and *Clauses*

In this section we show that for each commodity  $c_{lm}^{pv} \in \text{Constants}$  and for each commodity  $c_r^\omega \in \text{Clauses}$  the cycle time in an optimal solution is  $t_{c_{lm}^{pv}}^*$  and  $t_{c_r^\omega}^*$ , respectively, regardless of the cycle times of any other commodity in the problem.

For each commodity  $c_{lm}^{pv} \in \text{Constants}$  we define 2 EOQ problems. In the first EOQ problem, denoted  $\theta_1$ , we define:  $h_1 = h_{c_{lm}^{pv}}$  and  $K_1 = K_{c_{lm}^{pv}}$ . The solution for this problem defines a lower bound on the marginal average periodic cost of commodity  $c_{lm}^{pv}$  assuming no joint order costs are necessary.

**Lemma 2** *The integer optimal solution to  $\theta_1$  is  $t_{c_{lm}^{pv}}^*$ .*

**Proof.** According to Eq. (2), the optimal solution to the continuous  $\theta_1$  problem, denoted  $t_1^*$ , is:

$$t_1^* = \sqrt{\frac{2K_1}{2h_1}} = \sqrt{\frac{2K_{c_{lm}^{pv}}}{2h_{c_{lm}^{pv}}}} = \sqrt{\frac{K_{c_{lm}^{pv}}}{h_{c_{lm}^{pv}}}}. \quad (17)$$

Substituting for  $h_{c_{lm}^{pv}}$  and  $K_{c_{lm}^{pv}}$  using Eqs. (9) and (10) into Eq. (17), we get:

$$t_1^* = \sqrt{\frac{K_{c_{lm}^{pv}}}{h_{c_{lm}^{pv}}}} = \sqrt{\left(t_{c_{lm}^{pv}}^*\right)^2 - \frac{1}{2}}$$

since  $\sqrt{\left(t_{c_{lm}}^{pv}\right)^2 - \frac{1}{2}}$  is not an integer for any  $t_{c_{lm}}^{pv}$  that is an integer, the optimal solution will be defined according to the rounding rules in Eq. (3).

$$t_{c_{lm}}^{pv} - 1 < \sqrt{\left(t_{c_{lm}}^{pv}\right)^2 - \frac{1}{2}} < t_{c_{lm}}^{pv}.$$

Since

$$\sqrt{\left(t_{c_{lm}}^{pv} - 1\right) t_{c_{lm}}^{pv}} = \sqrt{\left(t_{c_{lm}}^{pv}\right)^2 - t_{c_{lm}}^{pv}} < \sqrt{\left(t_{c_{lm}}^{pv}\right)^2 - \frac{1}{2}}$$

the rounding rule states that the optimal solution will be  $\left\lfloor \sqrt{\left(t_{c_{lm}}^{pv}\right)^2 - \frac{1}{2}} \right\rfloor = t_{c_{lm}}^{pv}.$  ■

In the second EOQ problem, denoted  $\theta_2$ , we define:  $h_2 = h_{c_{lm}}^{pv}$  and  $K_2 = K_{c_{lm}}^{pv} + K_0$ . That is, we pay  $K_0$  for each order of commodity  $c_{lm}^{pv}$ . The solution for this problem defines an upper bound on the marginal average periodic cost of commodity  $c_{lm}^{pv}$ .

**Lemma 3** *The integer optimal solution to  $\theta_2$  is  $t_{c_{lm}}^{pv}$ .*

**Proof.** According to Eq. (2) the optimal solution to the continuous  $\theta_2$  problem, denoted  $t_2^*$ , is:

$$t_2^* = \sqrt{\frac{2K_2}{2h_2}} = \sqrt{\frac{2K_{c_{lm}}^{pv} + 2K_0}{2h_{c_{lm}}^{pv}}} = \sqrt{\frac{K_{c_{lm}}^{pv} + K_0}{h_{c_{lm}}^{pv}}}. \quad (18)$$

Substituting for  $h_{c_{lm}}^{pv}$ ,  $K_{c_{lm}}^{pv}$  and  $K_0$  using Eqs. (9), (10), and (16) into Eq. (18) we get:

$$t_2^* = \sqrt{\frac{K_{c_{lm}}^{pv} + K_0}{h_{c_{lm}}^{pv}}} = \sqrt{\left(t_{pv_{l,m}}^*\right)^2 + \frac{1}{2}}$$

since  $\sqrt{\left(t_{pv_{l,m}}^*\right)^2 + \frac{1}{2}}$  is not an integer for any  $t_{pv_{l,m}}^*$  that is an integer, the optimal solution will be defined according to the rounding rules in Eq. (3).

$$\begin{aligned} t_{c_{lm}}^{pv} &< \sqrt{\left(t_{c_{lm}}^{pv}\right)^2 + \frac{1}{2}} < t_{c_{lm}}^{pv} + 1 \\ \sqrt{\left(t_{c_{lm}}^{pv} + 1\right) t_{c_{lm}}^{pv}} &= \sqrt{\left(t_{c_{lm}}^{pv}\right)^2 + t_{c_{lm}}^{pv}} > \sqrt{\left(t_{c_{lm}}^{pv}\right)^2 + \frac{1}{2}} \end{aligned}$$

therefore according to the rounding rule, the optimal solution will be  $\left\lfloor \sqrt{\left(t_{c_{lm}}^{pv}\right)^2 + \frac{1}{2}} \right\rfloor = t_{c_{lm}}^{pv}.$  ■

**Theorem 1** *In any optimal solution to  $\Gamma$ , we have  $t_{c_{lm}}^{pv} = t_{c_{lm}}^{pv}$  for any  $c_{lm}^{pv} \in \text{Constants}$ .*

**Proof.** According to Lemmas 2 and 3 the solutions of  $P_1$  and  $P_2$  that define lower and upper bounds on  $\Delta_{c_{lm}}^{pv}(t_{c_{lm}}^{pv}, S)$ , respectively, are identical. Therefore, in any optimal solution the cycle time of commodity  $c_{lm}^{pv}$  is  $t_{c_{lm}}^{pv}$ . ■

Using Theorem 1 we can also learn about the cycle time of commodity  $c_r^\omega \in \text{Clauses}$  in an optimal solution to  $\Gamma$ .

**Theorem 2** *In any optimal solution to  $\Gamma$ , we have  $t_{c_r^\omega} = t_{c_r^\omega}^*$  for any  $c_r^\omega \in \text{Clauses}$ .*

**Proof.** Since the costs functions of commodity  $c_r^\omega \in \text{Clauses}$  in Eqs. (14) and (15) are identical to the costs functions of commodity  $c_{lm}^{pv} \in \text{Constants}$  in Eqs. (9) and (10), Theorem 1 holds for  $c_r^\omega \in \text{Clauses}$  as well. ■



### 3.2.2 Cycle time of commodities of type *Variables*

In this section we show that for each commodity  $c_i^x$  in *Variables* the cycle time in an optimal solution is either  $\underline{p}_i$  or  $\bar{p}_i$ . We denote by  $jr(t_{c_i^x}, S)$  the proportion of periods in which there is an order only of commodity  $t_{c_i^x}$ . Therefore,  $\Delta_{c_i^x}(t_{c_i^x}, S)$  is given by

$$\Delta_{c_i^x}(t_{c_i^x}, S) = \frac{K_{c_i^x}}{t_{c_i^x}} + t_{c_i^x} h_{c_i^x} + K_0 \cdot jr(t_{c_i^x}, S).$$

In order to analyze  $\Delta_{c_i^x}(t_{c_i^x}, S)$  we bound  $jr(t_{c_i^x}, S)$ . To do so we have to meticulously calculate the average periodic marginal addition of joint replenishment cost when choosing a cycle time of  $t_{c_i^x}$ . As we shall show next, the values in Eq. (4) – (8) are meaningful and where not chosen arbitrarily.

For any prime number  $p$ , the proportion of periods that are not a multiplication of  $p$  is

$$1 - \frac{1}{p} = \frac{p-1}{p}.$$

For any set of prime numbers  $A$ , the proportion of periods that are not multiples of any prime number  $p \in A$  is

$$\prod_{p \in A} \left( \frac{p-1}{p} \right).$$

Therefore, the proportion of periods that are not multiples of any prime number  $p_l \in PP$  is given by:

$$\prod_{p \in PP} \left( \frac{p-1}{p} \right) = \alpha_c.$$

Similarly,  $\alpha_v$ ,  $\bar{\alpha}_v$ ,  $\underline{\alpha}_v$  and  $\alpha_n$  represent the proportion of periods that are not multiples of any prime number  $p \in VP$ ,  $\bar{p}_i : c_i^x \in Variable$ ,  $\underline{p}_i : c_i^x \in Variable$  and  $p_{[i]} : i < n$ , respectively.

When calculating  $jr(t_c, S)$  we may first divide the time horizon by  $t_c$ , ending up with a new time horizon that represents only periods where  $c$  was actually ordered. Out of this new set of time periods, denoted by  $T_{t_c}$ , we try to account for the proportion of periods where  $c$  was ordered alone and actually initiated a joint replenishment that would not have been initiated otherwise. In other words, we are looking for the proportion of periods not covered by other orders within  $T_{t_c}$ . The frequency of ordering any two commodities at the same period is actually the least common denominator of each of their individual frequencies. For example, if commodities  $c_1$  and  $c_2$  have cycle times of 15 and 9 periods, respectively, they would be jointly ordered every 45 periods. If we were to consider  $T_{t_{c_1}}$ , which divides the time horizon by  $t_1 = 15$ , then  $1/3$  of the periods in  $T_{t_{c_1}}$  would have already been covered by  $c_2$ .

According to Theorem 1, in any optimal solution to  $\Gamma$   $t_{c_i^x} = t_{c_i^x}^*$  for any  $c_i^x \in Constants$ , and therefore we can consider only solutions in which  $t_{c_i^x} = t_{c_i^x}^*$  for any  $c_i^x \in Constants$ . Note that since there is a commodity  $c_i^x \in Constants$  for each combination of  $p_l \in PP$  and  $v_m \in VP$ , we can assume that at any period that is a multiplication of  $\underline{p}_i$  and  $p_l \in PP$  or a multiplication of  $\bar{p}_i$  and  $p_l \in PP$ , there is an order placed due to the commodities in *Constants*. Therefore  $jr(t_{c_i^x}, S)$  is not greater than the proportion of periods whose factors include  $t_{c_i^x}$  and exclude all prime numbers of set  $PP$ . This proportion is given by:

$$jr(t_{c_i^x}, S) < \frac{1}{t_{c_i^x}} \cdot \alpha_c.$$

Therefore,

$$UB\left(\Delta_{c_i^x}(t_{c_i^x} \in \{\underline{p}_i, \bar{p}_i\}, S)\right) = \frac{K_{c_i^x}}{t_{c_i^x}} + t_{c_i^x} h_{c_i^x} + K_0 \cdot \frac{1}{t_{c_i^x}} \cdot \alpha_c. \quad (19)$$

The lower bound on this marginal cost for an arbitrary cycle time  $t_{c_i^x}$  is given by the solution  $S$  in which there is a cycle time  $t \in S$  such that  $\text{mod}(t_i, t) = 0$ . In this case  $jr(t_{c_i^x}, S) = 0$  and then the lower bound is

given by:

$$LB(\Delta_{c_i^x}(t_{c_i^x})) = \frac{K_{c_i^x}}{t_{c_i^x}} + t_{c_i^x} h_{c_i^x}. \quad (20)$$

In order to prove that for each commodity  $c_i^x \in \text{Variables}$  the cycle time in an optimal solution is either  $\underline{p}_i$  or  $\bar{p}_i$ , we first show that the optimal solution is bounded by the range  $[\underline{p}_i, \bar{p}_i]$ .

**Claim 1** For each  $c_i^x \in \text{Variables}$  and for every solution  $S$ ,  $\Delta_{c_i^x}(\bar{p}_i, S) \leq \Delta_{c_i^x}(\bar{p}_i + 1, S)$

**Proof.** We show now that  $UB(\Delta_{c_i^x}(\bar{p}_i)) \leq LB(\Delta_{c_i^x}(\bar{p}_i + 1))$ . Using Eqs. (19) and (20) with  $t_{c_i^x} = \bar{p}_i$  and  $t_{c_i^x} = \bar{p}_i + 1$ , respectively, we get:

$$\begin{aligned} UB(\Delta_{c_i^x}(\bar{p}_i)) &= \frac{K_{c_i^x}}{\bar{p}_i} + \bar{p}_i h_{c_i^x} + K_0 \cdot \frac{1}{\bar{p}_i} \cdot \alpha_c \\ LB(\Delta_{c_i^x}(\bar{p}_i + 1)) &= \frac{K_{c_i^x}}{(\bar{p}_i + 1)} + (\bar{p}_i + 1) h_{c_i^x}. \end{aligned}$$

Therefore,

$$\begin{aligned} &LB(\Delta_{c_i^x}(\bar{p}_i + 1)) - UB(\Delta_{c_i^x}(\bar{p}_i)) \\ &= \frac{K_{c_i^x}}{(\bar{p}_i + 1)} + (\bar{p}_i + 1) h_{c_i^x} - \frac{K_{c_i^x}}{\bar{p}_i} - \bar{p}_i h_{c_i^x} - K_0 \cdot \frac{1}{\bar{p}_i} \cdot \alpha_c \\ &= h_{c_i^x} - \frac{K_{c_i^x}}{\bar{p}_i(\bar{p}_i + 1)} - K_0 \cdot \frac{1}{\bar{p}_i} \cdot \alpha_c. \end{aligned} \quad (21)$$

Substituting for  $\bar{p}_i = \underline{p}_i + b_i$  (see Condition 1) into Eq. (21) we get:

$$\begin{aligned} &LB(\Delta_{c_i^x}(\bar{p}_i + 1)) - UB(\Delta_{c_i^x}(\bar{p}_i)) \\ &= h_{c_i^x} - \frac{K_{c_i^x}}{(\underline{p}_i + b_i)(\underline{p}_i + b_i + 1)} - K_0 \cdot \frac{1}{(\underline{p}_i + b_i)} \cdot \alpha_c. \end{aligned} \quad (22)$$

Substituting for  $h_{c_i^x}$ ,  $K_{c_i^x}$ , and  $K_0$  using Eqs. (11), (12), and (16) into Eq. (22) we get:

$$\begin{aligned} &LB(\Delta_{c_i^x}(\bar{p}_i + 1)) - UB(\Delta_{c_i^x}(\bar{p}_i)) \\ &= h_{c_i^x} - \frac{h_{c_i^x} \cdot \underline{p}_i (\underline{p}_i + b_i)}{(\underline{p}_i + b_i)(\underline{p}_i + b_i + 1)} + \frac{\frac{\underline{p}_i + b_i}{\underline{p}_i + b_i - 1} \alpha_c \bar{\alpha}_v}{(\underline{p}_i + b_i)(\underline{p}_i + b_i + 1)} - \frac{1}{(\underline{p}_i + b_i)} \cdot \alpha_c \\ &= h_{c_i^x} \frac{b_i + 1}{(\underline{p}_i + b_i + 1)} + \frac{\alpha_c \bar{\alpha}_v}{(\underline{p}_i + b_i - 1)(\underline{p}_i + b_i + 1)} - \frac{1}{(\underline{p}_i + b_i)} \cdot \alpha_c \\ &= \alpha_c \left( \frac{(\underline{p}_i^2 - b_i^2)(b_i + 1)}{\underline{p}_i (\underline{p}_i + \frac{b_i}{2}) \frac{b_i}{2} (\underline{p}_i + b_i + 1)} + \frac{\bar{\alpha}_v}{(\underline{p}_i + b_i - 1)(\underline{p}_i + b_i + 1)} - \frac{1}{(\underline{p}_i + b_i)} \right) \\ &> \alpha_c \left( \frac{2(\underline{p}_i^2 - b_i^2)}{\underline{p}_i (\underline{p}_i + \frac{b_i}{2}) (\underline{p}_i + b_i + 1)} - \frac{1}{(\underline{p}_i + \frac{b_i}{2})} \right) \\ &= \alpha_c \frac{\underline{p}_i^2 - 2b_i^2 - \underline{p}_i b_i - \underline{p}_i}{\underline{p}_i (\underline{p}_i + \frac{b_i}{2}) (\underline{p}_i + b_i + 1)}. \end{aligned}$$

$\underline{p}_i$  is bounded below by  $n^{6\tilde{b}}$  (See Condition 3); thus, for an input  $n > 2$  the numerator is positive even for

the upper bound of 256 on  $b$  ([31]) and a lower bound of 2 on  $\tilde{b}$ ; thus, for any permissible  $\underline{p}_i$  :

$$LB(\Delta_{c_i^x}(\bar{p}_i + 1)) - UB(\Delta_{c_i^x}(\bar{p}_i)) > 0$$

■

**Claim 2** For each  $c_i^x \in \text{Variables}$  and for every solution  $S$ ,  $\Delta_{c_i^x}(\bar{p}_i, S) \leq \Delta_{c_i^x}(\underline{p}_i - 1, S)$

**Proof.** We show now that  $LB(\Delta_{c_i^x}(\underline{p}_i + b_i + 1)) \leq LB(\Delta_{c_i^x}(\underline{p}_i - 1))$ . Note that  $LB(\Delta_{c_i^x}(\underline{p}_i + b_i + 1))$  and  $LB(\Delta_{c_i^x}(\underline{p}_i - 1))$  are in fact the standalone costs  $g(\underline{p}_i + b_i + 1)$  and  $g(\underline{p}_i - 1)$ , respectively. According to the rounding rules in Eq. (3) :  $g(\underline{p}_i + b_i + 1) < g(\underline{p}_i - 1)$  if  $\underline{p}_i - 1 < t_{c_i^x}^* < \underline{p}_i + b_i + 1$  and

$$\sqrt{(\underline{p}_i + b_i + 1)(\underline{p}_i - 1)} < t_{c_i^x}^*.$$

The optimal solution to the standalone problem,  $t_{c_i^x}^*$  is given by:

$$\begin{aligned} \sqrt{\frac{K_{c_i^x}}{h_{c_i^x}}} &= \sqrt{\underline{p}_i(\underline{p}_i + b_i) - \frac{\frac{\underline{p}_i + b_i}{\underline{p}_i + b_i - 1} \alpha_c \bar{\alpha}_v}{h_{c_i^x}}} = \sqrt{\underline{p}_i(\underline{p}_i + b_i) - \frac{\frac{\underline{p}_i + b_i}{\underline{p}_i + b_i - 1} \bar{\alpha}_v}{\left(\frac{\underline{p}_i^2 - b_i^2}{\underline{p}_i(\underline{p}_i + \frac{b_i}{2})}\right)^{\frac{b_i}{2}}}} \\ &> \sqrt{\underline{p}_i(\underline{p}_i + b_i) - \frac{\underline{p}_i(\underline{p}_i + \frac{b_i}{2}) \frac{b_i}{2}}{(\underline{p}_i + b_i - 1)(\underline{p}_i - b_i)}} \\ &> \sqrt{\underline{p}_i(\underline{p}_i + b_i) - \frac{b_i \underline{p}_i}{2(\underline{p}_i - b_i)}} > \sqrt{\underline{p}_i^2 + b_i \underline{p}_i - b_i} \\ &> \sqrt{\underline{p}_i^2 + b_i \underline{p}_i - b_i - 1} = \sqrt{(\underline{p}_i + b_i + 1)(\underline{p}_i - 1)}. \end{aligned}$$

Therefore,  $LB(\Delta_{c_i^x}(\underline{p}_i + b_i + 1)) \leq LB(\Delta_{c_i^x}(\underline{p}_i - 1))$ . According to Claim 1  $\Delta_{c_i^x}(\bar{p}_i, S) \leq \Delta_{c_i^x}(\bar{p}_i + 1, S)$ ; hence,  $\Delta_{c_i^x}(\bar{p}_i, S) \leq \Delta_{c_i^x}(\bar{p}_i + 1, S) \leq \Delta_{c_i^x}(\underline{p}_i - 1, S)$ . ■

**Theorem 3** In any optimal solution to  $\Gamma$ ,  $t_{c_i^x} \in [\underline{p}_i, \bar{p}_i]$  for any  $c_i^x \in \text{Variables}$ .

**Proof.** According to Claims 1 and 2 for each variable  $c_i^x \in \text{Variable}$ ,  $UB(\Delta_{c_i^x}(\bar{p}_i)) \leq LB(\Delta_{c_i^x}(\bar{p}_i + 1)) \leq LB(\Delta_{c_i^x}(\underline{p}_i - 1))$ . Due to the convex nature of the cost function and since  $\underline{p}_i < t_{c_i^x}^* < \bar{p}_i$ ,  $LB(\Delta_{c_i^x}(\bar{p}_i + 1))$  is a lower bound on any solution  $t_{c_i^x} \notin [\underline{p}_i, \bar{p}_i]$ . Therefore in any optimal solution to  $\Gamma$ ,  $t_{c_i^x} \in [\underline{p}_i, \bar{p}_i]$ . ■

Accordingly, we consider only  $t_{c_i^x} \in \{\underline{p}_i + y : 0 \leq y \leq b_i\}$ . In the next claim we prove that any solution  $t_{c_i^x} \notin \{\underline{p}_i, \bar{p}_i\}$  is not the optimal solution. Let us assume that  $t_{c_i^x} \in \{\underline{p}_i + y : 0 < y < b_i\}$  and find a new lower bound for the solution. Note that  $t_{c_i^x}$  might not be a prime number.

Let us calculate the lower bound for  $jr(t_{c_i^x}, S)$ . According to Condition 4 none of the factors of  $t_{c_i^x}$  belong to  $VP$ . However, the factorials of  $t_{c_i^x}$  may include primes  $p \in PP$ . If that happens, Theorem 1 states that at each period that is a multiple of a prime number  $p \in VP$  and  $t_{c_j^x}$ , there is an order of another commodity  $c_{lm}^{pv} \in \text{Constants}$ . Thus, as a lower bound of  $jr(t_{c_i^x}, S)$ ,  $\frac{1}{\underline{p}_i + y}(1 - \alpha_v)$  of the periods may already be covered. Of the remaining  $\frac{1}{\underline{p}_i + y} \alpha_v$  periods there might be periods covered by some other  $c_j^x \in \text{Variables}$  sharing the

same factorials with  $t_{c_i^x}$ . Any two commodities  $c_i^x, c_j^x \in \text{Variables}$  with  $t_{c_i^x}$  and  $t_{c_j^x}$  that are not primes, might share some common prime factors as well as a unique multiplier (might not be a prime one) of each one. Hence, we may represent their respective cycle times by  $t_{c_i^x} = \eta\mu_i$  and  $t_{c_j^x} = \eta\mu_j$ , where  $\mu_i$  and  $\mu_j$  represent the unique elements of  $t_{c_i^x}$ , and  $t_{c_j^x}$  and  $\eta$  and represent their common factors.

Calculating  $jr(t_{c_i^x}, S)$ , each period in  $T_{t_{c_i^x}}$  that is a multiple of  $\mu_j$  is covered by  $c_j^x$ . Since there are  $n$  commodities of type  $c_i^x \in \text{Variables}$ , there are at most  $n-1$  such unique  $\mu_j$  elements. The more common factors these elements share and the smaller they are the more time periods they will cover in  $T_{t_{c_i^x}}$ . Accordingly, a lower bound for  $jr(\underline{p}_i + y, S)$  considers the  $n$  smallest primes as  $\mu_j$  values.

$$jr(t_{c_i^x}, S) > \frac{1}{t_{c_i^x}} \left( \alpha_v \prod_{\substack{q[j] \in PP \\ j < n}} \left( 1 - \frac{1}{q[j]} \right) \right) = \frac{1}{t_{c_i^x}} (\alpha_v a_n).$$

Hence, assuming the optimal cycle time according to Eq. (2):

$$LB(\Delta_{c_i^x}(t_{c_i^x} \notin \{\underline{p}_i, \bar{p}_i\})) = \frac{K_{c_i^x} + K_0(\alpha_v a_n)}{t_{c_i^x}} + h_{c_i^x}(t_{c_i^x}). \quad (23)$$

In the next Lemmas we show that for each  $c_i^x \in A$  and for every solution  $S$  the optimal  $t_{c_i}$  is either  $\underline{p}_i$  or  $\bar{p}_i$ .

**Claim 3** For each  $c_i^x \in \text{Variables}$  and for every solution  $S$ ,  $\Delta_{c_i^x}(\underline{p}_i, S) < \Delta_{c_i^x}(\underline{p}_i + y, S)$  for  $0 < y < b_i$ .

**Proof.** We show now that  $UB(\Delta_{c_i^x}(\underline{p}_i)) < LB(\Delta_{c_i^x}(\underline{p}_i + y))$ . Using Eq. (23) with  $t_{c_i^x} = \underline{p}_i + y$  and Eq. (19) we get:

$$\begin{aligned} & LB(\Delta_{c_i^x}(\underline{p}_i + y)) - UB(\Delta_{c_i^x}(\underline{p}_i)) \\ &= \frac{K_{c_i^x} + K_0(\alpha_v a_n)}{\underline{p}_i + y} + h_{c_i^x}(\underline{p}_i + y) - \left( \frac{K_{c_i^x}}{\underline{p}_i} + \underline{p}_i h_{c_i^x} + K_0 \cdot \frac{\alpha_c}{\underline{p}_i} \right) \\ &= \frac{-yK_{c_i^x}}{\underline{p}_i(\underline{p}_i + y)} + y h_{c_i^x} + \frac{\alpha_v a_n}{\underline{p}_i + y} - \frac{\alpha_c}{\underline{p}_i} \end{aligned} \quad (24)$$

Substituting for  $h_{c_i^x}$ ,  $K_{c_i^x}$  and  $K_0$  using Eqs. (11), (12) and (16) into Eq. (24) we get:

$$\begin{aligned} & LB(\Delta_{c_i^x}(\underline{p}_i + y)) - UB(\Delta_{c_i^x}(\underline{p}_i)) \\ &= \frac{-y h_{c_i^x} \cdot \underline{p}_i(\underline{p}_i + b_i) + y \frac{\underline{p}_i + b_i}{\underline{p}_i + b_i - 1} \alpha_c \bar{\alpha}_v}{\underline{p}_i(\underline{p}_i + y)} + y h_{c_i^x} + \frac{\alpha_v a_n}{\underline{p}_i + y} - \frac{\alpha_c}{\underline{p}_i} \\ &= -y h_{c_i^x} \frac{b_i - y}{\underline{p}_i + y} + \frac{y(\underline{p}_i + b_i) \alpha_c \bar{\alpha}_v}{\underline{p}_i(\underline{p}_i + y)(\underline{p}_i + b_i - 1)} + \frac{\alpha_v a_n}{\underline{p}_i + y} - \frac{\alpha_c}{\underline{p}_i} \\ &= -\alpha_c y \frac{\underline{p}_i^2 - b_i^2}{\underline{p}_i(\underline{p}_i + \frac{b_i}{2}) \frac{b_i}{2} \underline{p}_i + y} + \frac{y(\underline{p}_i + b_i) \alpha_c \bar{\alpha}_v}{\underline{p}_i(\underline{p}_i + y)(\underline{p}_i + b_i - 1)} + \frac{\alpha_v a_n}{\underline{p}_i + y} - \frac{\alpha_c}{\underline{p}_i} \\ &= \frac{-\alpha_c}{\underline{p}_i + y} \left( y \frac{\underline{p}_i^2 - b_i^2}{\underline{p}_i(\underline{p}_i + \frac{b_i}{2}) \frac{b_i}{2}} (b_i - y) - \frac{y(\underline{p}_i + b_i) \bar{\alpha}_v}{\underline{p}_i(\underline{p}_i + b_i - 1)} + \frac{(\underline{p}_i + y)}{\underline{p}_i} \right) + \frac{\alpha_v a_n}{\underline{p}_i + y} \\ &> \frac{-\alpha_c}{\underline{p}_i + y} \left( \frac{2y(b_i - y)}{b_i} + 2 \right) + \frac{\alpha_v a_n}{\underline{p}_i + y} = \frac{a_n}{\underline{p}_i + y} \left( \alpha_v - 2 \frac{\alpha_c}{a_n} \left( \frac{y(b_i - y)}{b_i} + 1 \right) \right). \end{aligned}$$

The value  $y(b_i - y)$  is maximized when  $y = 0.5b_i$ ; thus:

$$LB\left(\Delta_{c_i^x}\left(\underline{p}_i + y\right)\right) - UB\left(\Delta_{c_i^x}\left(\underline{p}_i\right)\right) > \frac{a_n}{\underline{p}_i + y} \left(\alpha_v - \frac{\alpha_c}{a_n} \left(\frac{b_i}{2} + 2\right)\right) \quad (25)$$

According to Ribenboim [35] and Condition 3 there are at least

$$\frac{0.91 \cdot n^{6\tilde{b}}}{6\tilde{b} \log n} > 3n$$

prime numbers in  $PP$ . Moreover,  $\alpha_c$  and  $a_n$  share the first  $n$  elements of their respective multiples. Therefore, we can cancel out these  $n$  elements and explicitly write  $\frac{\alpha_c}{a_n}$  in Eq. (25) as follows:

$$\begin{aligned} & LB\left(\Delta_{c_i^x}\left(\underline{p}_i + y\right)\right) - UB\left(\Delta_{c_i^x}\left(\underline{p}_i\right)\right) \\ & > \frac{a_n}{\underline{p}_i + 1} \left( \alpha_v - \left(\frac{b_i}{2} + 2\right) \prod_{\substack{j \geq n \\ p_{[j]} \leq n^{6\tilde{b}}}} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right) \right) \\ & = \frac{a_n}{\underline{p}_i + 1} \left( \alpha_v - \left(\frac{b_i}{2} + 2\right) \prod_{n \leq j \leq 3n} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right) \prod_{\substack{j \geq 3n \\ p_{[j]} \leq n^{6\tilde{b}}}} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right) \right). \end{aligned}$$

Note that both  $\alpha_v$  and  $\prod_{n \leq j \leq 3n} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right)$  are multiples of  $2n$  elements where each element in  $\alpha_v$  is bigger than each element in  $\prod_{n \leq j \leq 3n} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right)$  and therefore  $\alpha_v > \prod_{n \leq j \leq 3n} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right)$ . Hence,

$$\begin{aligned} & LB\left(\Delta_{c_i^x}\left(\underline{p}_i + y\right)\right) - UB\left(\Delta_{c_i^x}\left(\underline{p}_i\right)\right) \\ & > \frac{a_n \alpha_v}{\underline{p}_i + 1} \left( 1 - \left(\frac{b_i}{2} + 2\right) \prod_{\substack{3n < j \\ p_{[j]} \leq n^{6\tilde{b}}}} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right) \right). \end{aligned}$$

Using the upper bound of 256 on  $b_i$  (see [31]) we have:

$$\begin{aligned} & LB\left(\Delta_{c_i^x}\left(\underline{p}_i + y\right)\right) - UB\left(\Delta_{c_i^x}\left(\underline{p}_i\right)\right) \\ & > \frac{a_n \alpha_v}{\underline{p}_i + 1} \left( 1 - 130 \prod_{\substack{3n < j \\ p_{[j]} \leq n^{6\tilde{b}}}} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right) \right). \end{aligned} \quad (26)$$

In order to lower bound  $p_{[3n]}$  we use the bound presented in [35]:

$$p_{[3n]} > 0.91 \cdot 3n \ln(3n).$$

Substituting this bound into Eq. (26) we get:

$$\begin{aligned} & LB\left(\Delta_{c_i^x}\left(\underline{p}_i + y\right)\right) - UB\left(\Delta_{c_i^x}\left(\underline{p}_i\right)\right) \\ & > \frac{a_n \alpha_v}{\underline{p}_i + 1} \left( 1 - 130 \prod_{0.91 \cdot 3n \ln 3n < p_{[j]} \leq n^{6\tilde{b}}} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right) \right). \end{aligned} \quad (27)$$

According to Merten's theorems [21]:

$$\prod_{j \leq G} \left( \frac{p_{[j]} - 1}{p_{[j]}} \right) = \frac{e^{-\gamma \rho(G)}}{\ln(G)},$$

where  $0 < \rho(G) < \frac{4}{\ln(G+1)} + \frac{2}{G \ln(G)} + \frac{1}{2G}$  and  $\gamma$  is the Euler–Mascheroni constant. Hence,

$$\prod_{G' < j \leq G} \left( \frac{p_{[j]} - 1}{p_{[j]}} \right) = \frac{\frac{e^{-\gamma \rho(G)}}{\ln(G)}}{\frac{e^{-\gamma \rho(G')}}{\ln(G')}} = \frac{\ln(G') e^{\gamma(\rho(G') - \rho(G))}}{\ln(G)}.$$

Therefore, the bound on  $\prod_{0.91 \cdot 3n \ln 3n < p_{[j]} \leq n^{6\tilde{b}}} \left( \frac{p_{[j]} - 1}{p_{[j]}} \right)$  is given by:

$$\begin{aligned} & \prod_{0.91 \cdot 3n \ln 3n < p_{[j]} \leq n^{6\tilde{b}}} \left( \frac{p_{[j]} - 1}{p_{[j]}} \right) \\ &= \frac{\ln(0.91 \cdot 3n \ln 3n) e^{\gamma(\rho(0.91 \cdot 3n \ln 3n) - \rho(n^{6\tilde{b}}))}}{\ln(n^{6\tilde{b}})} \\ &< \frac{\ln(0.91 \cdot 3n \ln 3n)}{6\tilde{b} \ln n} e^{\gamma\left(\frac{4}{\ln(0.91 \cdot 3n \ln 3n + 1)} + \frac{2}{0.91 \cdot 3n \ln 3n \ln(0.91 \cdot 3n \ln 3n)} + \frac{1}{2 \cdot 0.91 \cdot 3n \ln 3n}\right)}. \end{aligned} \quad (28)$$

The function in Eq. (28) is a multiple of 2 positive non-increasing functions of  $n > 1$ :  
Functions

$$\frac{\ln(0.91 \cdot 3n \ln 3n)}{6\tilde{b} \ln n}$$

and

$$e^{\gamma\left(\frac{4}{\ln(0.91 \cdot 3n \ln 3n + 1)} + \frac{2}{0.91 \cdot 3n \ln 3n \ln(0.91 \cdot 3n \ln 3n)} + \frac{1}{2 \cdot 0.91 \cdot 3n \ln 3n}\right)}.$$

Therefore, the function in Eq. (28) is a non-increasing function of  $n$  for  $n > 1$ . For  $n = 64$  the function in (28) is smaller than  $\frac{1}{130}$  and therefore, for any  $n \geq 64$  we can substitute the upper bound of  $\frac{1}{130}$  into Eq. (27):

$$\begin{aligned} & LB\left(\Delta_{c_i^x}(\underline{p}_i + y)\right) - UB\left(\Delta_{c_i^x}(\underline{p}_i)\right) \\ &> \frac{a_n \alpha_v}{\underline{p}_i + 1} \left(1 - 130 \prod_{0.91 \cdot 3n \ln 3n < p_{[j]} \leq n^{6\tilde{b}}} \left(\frac{p_{[j]} - 1}{p_{[j]}}\right)\right) \\ &> \frac{a_n \alpha_v}{\underline{p}_i + 1} (1 - 1) = 0. \end{aligned}$$

■

Figure 2 illustrates the behavior of the lower and upper bounds on  $\Delta_{c_i^x}(t_{c_i^x})$  within the range  $[\underline{p}_i - 2, \bar{p}_i + 2]$ . We arbitrarily chose to show the bounds for  $b_i = 2$ . The lower bound for  $\underline{p}_i - 1$  and  $\bar{p}_i + 1$  is their standalone average cost (depicted by the light blue line). However,  $\underline{p}_i + y$  for  $0 < y < b_i$  requires a tighter bound in order to disprove its optimality (depicted on the pink line).

**Theorem 4** In any optimal solution to  $\Gamma$ ,  $t_{c_i^x} \in \{\underline{p}_i, \bar{p}_i\}$  for any  $c_i^x \in \text{Variables}$ .

**Proof.** According to Theorem 4 the range of the optimal solution for commodity  $c_i^x$  is  $t_{c_i^x} \in [\underline{p}_i, \bar{p}_i]$ . According to Claim 3 the solution  $t_{c_i^x} = \underline{p}_i + y$  for  $0 < y < b_i$  costs more than the solution upper bound on  $t_{c_i^x} \in \{\underline{p}_i, \bar{p}_i\}$ . Therefore, in any optimal solution to  $\Gamma$ ,  $t_{c_i^x} \in \{\underline{p}_i, \bar{p}_i\}$ . ■

Figure 2: Lower and upper bounds for  $\Delta_{c_i^x}(t_{c_i^x})$  in the range  $t_{c_i^x} \in [\underline{p}_i - 1, \bar{p}_i + 1]$ , depicted for  $b_i = 2$ .

### 3.2.3 Proof that solving $\Gamma$ optimally is equivalent to solving $\varphi$

In this section we show that an optimal solution to  $\Gamma$  defines an assignment  $\alpha$  to  $\varphi$ . First we define an assignment  $\alpha$  given an optimal solution  $S$  to  $\Gamma$  as follows: for each commodity  $c_i^x \in \text{Variables}$  if the cycle time  $t_{c_i^x} = \underline{p}_i$  set  $\alpha(x_i) = \text{false}$ . Otherwise, if the cycle time  $t_{c_i^x} = \bar{p}_i$ , set  $\alpha(x_i) = \text{true}$ . Note that according to Theorem 4  $t_{c_i^x} \in \{\underline{p}_i, \bar{p}_i\}$ ; therefore, these are the only options. We now want to show that if  $\varphi$  is satisfiable then assignment  $\alpha$  that satisfies  $\varphi$  gives a solution to  $\Gamma$  that is lower than any solution  $\alpha'$  that doesn't satisfy  $\varphi$ . Thus by minimizing  $\Gamma$  we solve  $\varphi$ .

In order to do so we define 3 sets of periods. The first set, denoted by  $T^{\text{Constants}}$ , includes all the periods in which there is an order of at least one commodity  $c_{lm}^{pv} \in \text{Constants}$ . The second set, denoted by  $T^{\text{Variables}}$ , includes all the periods in which there is an order of at least one commodity  $c_i^x \in \text{Variables}$ . The third set, denoted by  $T^{\text{Clauses}}$ , includes all the periods in which there is an order of at least one commodity  $c_r^\omega \in \text{Clauses}$ . Accordingly, we formulate the total cost of solution  $S$ , denoted by  $TC(S)$ , as a sum of 3 cost functions: The first cost function,  $TC_{\text{Constants}}(S)$ , sums all the costs that are associated with the commodities  $c_{lm}^{pv} \in \text{Constants}$ , including all the joint replenishment costs at periods  $t \in T^{\text{Constants}}$ . The second cost function,  $TC_{\text{Variables}}(S)$ , sums all the costs that are associated with the commodities  $c_i^x \in \text{Variables}$ , including all the joint replenishment costs at periods  $t \in T^{\text{Variables}} \setminus T^{\text{Constants}}$ . The third cost function,  $TC_{\text{Clauses}}(S)$ , sums all the costs that are associated with the commodities  $c_r^\omega \in \text{Clauses}$ , including all the joint replenishment costs at periods  $t \in T^{\text{Clauses}} \setminus (T^{\text{Variables}} \cup T^{\text{Constants}})$ . Note that

$$\begin{aligned} & T^{\text{Constants}} \cup (T^{\text{Variables}} \setminus T^{\text{Constants}}) \cup (T^{\text{Clauses}} \setminus (T^{\text{Variables}} \cup T^{\text{Constants}})) \\ = & T^{\text{Constants}} \cup T^{\text{Variables}} \cup T^{\text{Clauses}} \end{aligned}$$

and

$$T^{\text{Constants}} \cap (T^{\text{Variables}} \setminus T^{\text{Constants}}) \cap (T^{\text{Clauses}} \setminus (T^{\text{Variables}} \cup T^{\text{Constants}})) = \emptyset.$$

Therefore,

$$TC(S) = TC_{\text{Constants}}(S) + TC_{\text{Variables}}(S) + TC_{\text{Clauses}}(S).$$

According to Theorem 1 the cost  $TC_{Constants}(S)$  is identical for any optimal solution to  $\Gamma$ .

$$TC_{Constants}(S) = \sum_{c_{lm}^{pv} \in Constants} \left( \frac{K_{c_{lm}^{pv}}}{t_{c_{lm}^{pv}}^*} + t_{c_{lm}^{pv}}^* \cdot h_{c_{lm}^{pv}} \right) + K_0 \cdot (1 - \alpha_c).$$

We now bound the cost function  $TC_{Variables}(S)$ . We denote  $\Delta_{c_i^x}^{TC_{Variables}}(t_{c_i^x}, S)$  as the marginal average periodic cost of the function  $TC_{Variables}(S)$  associated with commodity  $c$ 's cycle time  $t_{c_i^x}$ , where  $c_i^x \in Variables$ ,  $t_{c_i^x} \in \{\underline{p}_i, \bar{p}_i\}$  and a solution  $S$  that applies the characteristics of an optimal solution in Theorems 1-4 to the other commodities in the system. In the next claim we formulate bounds on  $\Delta_{c_i^x}^{TC_{Variables}}(t_{c_i^x}, S)$ . Note that

$$\Delta_{c_i^x}^{TC_{Variables}}(t_{c_i^x}, S) = \frac{K_{c_i^x}}{t_{c_i^x}} + t_{c_i^x} h_{c_i^x} + K_0 \cdot jr^{TC_{Variables}}(t_{c_i^x}, S)$$

where  $jr^{TC_{Variables}}(t_{c_i^x}, S)$  is the proportion of periods in  $T^{Variables} \setminus T^{Constants}$  in which there is an order only of commodity  $t_{c_i^x}$ . According to Lemma 1 the cycle time  $t_{c_i^x}$  is not a multiple of any other cycle time  $t_{c_j^x}$  for any  $c_j^x \in Variables$ . Note that cycle time  $t_{c_j^x}$  is a prime number for any  $c_j^x \in Variables$  and therefore  $t_{c_i^x}$  and  $t_{c_j^x}$  do not share factors. According to Theorem 1 there is no commodity with a cycle time that is a factor of  $t_{c_i^x}$  in  $Constants$ . However, for each  $t_{c_j^x} \in \{\underline{p}_j, \bar{p}_j\}$  and a prime number  $p \in PP$ , there is a commodity  $c_{lm}^{pv} \in Constants$  with a cycle time  $c_{lm}^{pv} = p \cdot t_{c_j^x}$ , which means that at least  $\alpha_c = \prod_{p \in PP} \left(\frac{p-1}{p}\right)$  of the periods in  $TC_{Variables}$  associated with  $t_{c_i^x}$  are covered by  $T^{Constants}$ . Therefore, for a solution  $t_{c_i^x} \in \{\underline{p}_i, \bar{p}_i\}$ ,  $jr^{TC_{Variables}}(t_{c_i^x}, S)$  is given by:

$$jr^{TC_{Variables}}(t_{c_i^x}, S) = \frac{1}{t_{c_i^x}} \cdot \alpha_c \prod_{i \neq j, t_{c_j^x} \in S} \left( \frac{t_{c_j^x} - 1}{t_{c_j^x}} \right). \quad (29)$$

Therefore

$$\Delta_{c_i^x}^{TC_{Variables}}(t_{c_i^x}, S) = \frac{K_{c_i^x}}{t_{c_i^x}} + t_{c_i^x} h_{c_i^x} + K_0 \cdot \frac{1}{t_{c_i^x}} \cdot \alpha_c \prod_{i \neq j, t_{c_j^x} \in S} \left( \frac{t_{c_j^x} - 1}{t_{c_j^x}} \right). \quad (30)$$

**Claim 4** For each  $c_i^x \in Variables$ , and for each optimal solution  $S$ ,  $\Delta_{c_i^x}^{TC_{Variables}}(\underline{p}_i, S) \leq \Delta_{c_i^x}^{TC_{Variables}}(\bar{p}_i, S)$ .

**Proof.** Using Eq. (30) we get:

$$\begin{aligned} \Delta_{c_i^x}^{TC_{Variables}}(\underline{p}_i, S) &= \frac{K_{c_i^x}}{\underline{p}_i} + \underline{p}_i h_{c_i^x} + K_0 \cdot \frac{1}{\underline{p}_i} \cdot \alpha_c \prod_{j \neq i, t_{c_j^x} \in S} \left( \frac{t_{c_j^x} - 1}{t_{c_j^x}} \right); \\ \Delta_{c_i^x}^{TC_{Variables}}(\bar{p}_i, S) &= \frac{K_{c_i^x}}{\bar{p}_i} + \bar{p}_i h_{c_i^x} + K_0 \cdot \frac{1}{\bar{p}_i} \cdot \alpha_c \prod_{j \neq i, t_{c_j^x} \in S} \left( \frac{t_{c_j^x} - 1}{t_{c_j^x}} \right). \end{aligned}$$

Substituting for  $\bar{p}_i = \underline{p}_i + b_i$  we get:

$$\Delta_{c_i^x}^{TC_{Variables}}(\bar{p}_i, S) = \frac{K_{c_i^x}}{(\underline{p}_i + b_i)} + (\underline{p}_i + b_i) h_{c_i^x} + K_0 \cdot \frac{1}{(\underline{p}_i + b_i)} \cdot \alpha_c \prod_{j \neq i, t_{c_j^x} \in S} \left( \frac{t_{c_j^x} - 1}{t_{c_j^x}} \right).$$

Therefore

$$\begin{aligned} &\Delta_{c_i^x}^{TC_{Variables}}(\bar{p}_i, S) - \Delta_{c_i^x}^{TC_{Variables}}(\underline{p}_i, S) \\ &= b_i \left( h_{c_i^x} - \frac{K_{c_i^x}}{\underline{p}_i (\underline{p}_i + b_i)} - \frac{K_0}{\underline{p}_i (\underline{p}_i + b_i)} \cdot \alpha_c \prod_{j \neq i, t_{c_j^x} \in S} \left( \frac{t_{c_j^x} - 1}{t_{c_j^x}} \right) \right). \end{aligned} \quad (31)$$



Substituting for  $h_{c_i^x}$ ,  $K_{c_i^x}$  and  $K_0$  using Eqs. (11), (12), and (16) into Eq. (31), we get:

$$\begin{aligned} & \Delta_{c_i^x}^{TC_{Variables}}(\bar{p}_i, S) - \Delta_{c_i^x}^{TC_{Variables}}(\underline{p}_i, S) \\ &= b_i \left( \frac{\alpha_c \bar{\alpha}_v}{\underline{p}_i (\underline{p}_i + b_i - 1)} - \frac{1}{\underline{p}_i (\underline{p}_i + b_i)} \cdot \alpha_c \prod_{j \neq i, t_{c_j^x} \in S} \left( \frac{t_{c_j^x} - 1}{t_{c_j^x}} \right) \right). \end{aligned} \quad (32)$$

Recall that  $\forall j : \underline{p}_j \leq t_{c_j^x} \leq \bar{p}_j$ ; thus,

$$\forall j : \frac{\underline{p}_j - 1}{\underline{p}_j} \leq \frac{t_{c_j^x} - 1}{t_{c_j^x}} \leq \frac{\bar{p}_j - 1}{\bar{p}_j}.$$

Therefore,

$$\begin{aligned} \alpha_v \cdot \frac{\underline{p}_i}{\underline{p}_i - 1} &= \prod_{i \neq j, c_j^x \in Variable} \left( \frac{\underline{p}_j - 1}{\underline{p}_j} \right) \\ &\leq \prod_{j \neq i, t_{c_j^x} \in S} \left( \frac{t_{c_j^x} - 1}{t_{c_j^x}} \right) \\ &\leq \prod_{i \neq j, c_j^x \in Variable} \left( \frac{\bar{p}_j - 1}{\bar{p}_j} \right) = \bar{\alpha}_v \cdot \frac{\underline{p}_i + b_i}{\underline{p}_i + b_i - 1}, \end{aligned}$$

Substituting for  $\prod_{j \neq i, t_{c_j^x} \in S} \left( \frac{t_{c_j^x} - 1}{t_{c_j^x}} \right)$  into Eq. (32) we get

$$\begin{aligned} & \Delta_{c_i^x}^{TC_{Variables}}(\bar{p}_i, S) - \Delta_{c_i^x}^{TC_{Variables}}(\underline{p}_i, S) \\ &\geq b_i \left( \frac{\alpha_c \bar{\alpha}_v}{\underline{p}_i (\underline{p}_i + b_i - 1)} - \frac{1}{\underline{p}_i (\underline{p}_i + b_i)} \cdot \alpha_c \bar{\alpha}_v \cdot \frac{\underline{p}_i + b_i}{\underline{p}_i + b_i - 1} \right) = 0. \end{aligned}$$

■

We can now lower bound  $TC_{Variables}(S)$  by the solution in which  $\forall c_j^x \in Variables: t_{c_j^x} = \underline{p}_j$ . Similarly we can upper bound  $TC_{Variables}(S)$  by the solution in which  $\forall c_j^x \in Variables: t_{c_j^x} = \bar{p}_j$ . The costs of the lower and upper bounds on  $TC_{Variables}(S)$  are given by:

$$UB(TC_{Variables}) = \sum_{c_i^x \in Variables} \left( \frac{K_{c_i^x}}{\bar{p}_i} + \bar{p}_i \cdot h_{c_i^x} \right) + K_0 \cdot \alpha_c \cdot (1 - \bar{\alpha}_v); \quad (33)$$

$$LB(TC_{Variables}) = \sum_{c_i^x \in Variables} \left( \frac{K_{c_i^x}}{\underline{p}_i} + \underline{p}_i \cdot h_{c_i^x} \right) + K_0 \cdot \alpha_c \cdot (1 - \underline{\alpha}_v). \quad (34)$$

Last, according to Theorem 2, the cycle time of any commodity  $c_r^\omega \in Clauses$  where  $\omega_r = (z_i \cup z_j \cup z_s)$  is  $t_{c_r^\omega}^* = P(z_i) \cdot P(z_j) \cdot P(z_s)$  and  $P(z_i), P(z_j), P(z_s)$  are prime numbers; thus the only factors of  $t_{c_r^\omega}^*$  are  $P(z_i), P(z_j), P(z_s)$ . Moreover, the cycle times of the commodities  $c_r^\omega \in Clauses$  are not a multiple of one another, nor are they a multiple of any cycle time of any commodity  $c_{lm}^{pv} \in Constants$ .

We examine 2 scenarios. In the first there is a commodity with a cycle time that is a factor of  $t_{c_r^\omega}^*$ . Without loss of generality assume that there is a commodity with cycle time  $P(z_i)$ . Note that according to Theorems 1, 2, and 4 the only commodity that might have a cycle time of  $P(z_i)$  in an optimal solution is commodity  $c_i^x$ . If commodity  $c_i^x$  has a cycle time of  $P(z_i)$ , then in the assignment  $\alpha$  the value of the literal  $z_i$  is true. In this case the clause  $\omega_r = (z_i \cup z_j \cup z_s)$  is satisfied under  $\alpha$ . Note that if there is another commodity with cycle time that is a factor of  $t_{c_r^\omega}^*$ , then there will be no additional cost for the joint

replenishment. In the second scenario there is no commodity with a cycle time that is a factor of  $t_{c_r^\omega}^*$ . In this case we know that the clause  $\omega_r = (z_i \cup z_j \cup z_s)$  is unsatisfied under  $\alpha$ . To lower bound the marginal joint replenishment cost we perform a similar analysis to the one in Eq. (29). Yielding the proportion of periods in  $TClauses \setminus (TVariables \cup TConstants)$  in which there is an order only of commodity  $c_r^\omega$ , given by  $jr^{TClauses}(t_{c_r^\omega}, S)$

$$jr^{TClauses}(t_{c_r^\omega}, S) = \alpha_c \cdot \prod_{c_i^x \in Variable} \left( \frac{t_{c_i^x} - 1}{t_{c_i^x}} \right)$$

where  $t_{c_i^x}$  is the cycle time of commodity  $c_i^x$  in the solution  $S$ . Since  $c_i^x \in Variable : t_{c_i^x} \in \{\underline{p}_i, \bar{p}_i\}$ ,  $t_{c_i^x}$  is minimal at  $t_{c_i^x} = \underline{p}_i$ , and therefore a lower bound on the marginal joint replenishment cost for any optimal solution is

$$LB(jr^{TClauses}(t_{c_r^\omega})) = \alpha_c \cdot \underline{\alpha}_v.$$

We denote the group of all the clauses that are not satisfied under  $\alpha$  by  $F$ . We can now formulate the cost  $TClauses(S)$  as:

$$\begin{aligned} TClauses(S) = & \sum_{c_r^\omega \in Clauses} \left( \frac{K_{c_r^\omega}}{t_{c_r^\omega}^*} + t_{c_r^\omega}^* \cdot h_{c_r^\omega} \right) + \\ & K_0 \cdot \alpha_c \cdot \prod_{c_i^x \in Variable} \left( \frac{t_{c_i^x} - 1}{t_{c_i^x}} \right) \cdot \left( 1 - \prod_{c_r^\omega \in F} \frac{t_{c_r^\omega}^* - 1}{t_{c_r^\omega}^*} \right) \end{aligned} \quad (35)$$

(Note that by definition, the product of an empty set equals 1). The lower bound on  $TClauses(S)$  is given by:

$$\begin{aligned} LB(TClauses(S)) = & \sum_{c_r^\omega \in Clauses} \left( \frac{K_{c_r^\omega}}{t_{c_r^\omega}^*} + t_{c_r^\omega}^* \cdot h_{c_r^\omega} \right) + \\ & K_0 \cdot \alpha_c \cdot \underline{\alpha}_v \cdot \left( 1 - \prod_{c_r^\omega \in F} \frac{t_{c_r^\omega}^* - 1}{t_{c_r^\omega}^*} \right). \end{aligned} \quad (36)$$

Eq. (36) grows with the number of unsatisfied clauses; thus, in order to show that if  $\varphi$  is satisfiable then any solution that doesn't satisfy  $\varphi$  costs more than a solution that does, it is sufficient to show that the lower bound on a solution  $S$  in which there is only one unsatisfied clause costs more than the upper bound on a solution  $S'$  that satisfies all the clauses. Without loss of generality assume that the unsatisfied clause is  $\omega_r$ .

$$\begin{aligned} LB(TC(S)) &= TCConstants(S) + LB(TCVariables(S)) + LB(TC_{Clauses}(S)); \\ UB(TC(S')) &= TCConstants(S') + UB(TCVariables) + TC_{Clauses}(S'). \end{aligned}$$

Note that  $TCConstants(S) = TCConstants(S')$  is a constant unaffected by the assignment  $\alpha$  and that under the assumption that  $S'$  satisfies  $\varphi$  so does  $TC_{Clauses}(S')$ . For the remaining cost elements we use upper and lower bounds. We now show that  $UB(TC(S')) < LB(TC(S))$ .

$$\begin{aligned} & LB(TC(S)) - UB(TC(S')) \\ &= TCConstants(S) + LB(TCVariables(S)) + LB(TC_{Clauses}(S)) \\ &\quad - TCConstants(S') - UB(TCVariables) - TC_{Clauses}(S') \\ &= LB(TCVariables) - UB(TCVariables) + LB(TC_{Clauses}(S)) - TC_{Clauses}(S') > 0. \end{aligned} \quad (37)$$

In order to analyze the expression in Eq. (37), we prove the following Claims:

**Claim 5**  $LB(TCVariables) - UB(TCVariables) > -b_i^2 \alpha_c \bar{\alpha}_v \underline{p}_1^{\frac{1}{3b}-4}$

**Proof.** According to Eqs. (33) and (34) :

$$\begin{aligned}
& LB(TC_{Variables}) - UB(TC_{Variables}) \\
&= \sum_{c_i^x \in Variables} \left( \frac{K_{c_i^x}}{\underline{p}_i} + \underline{p}_i \cdot h_{c_i^x} \right) + K_0 \cdot \alpha_c \cdot (1 - \underline{\alpha}_v) \\
&\quad - \sum_{c_i^x \in Variables} \left( \frac{K_{c_i^x}}{\bar{p}_i} + \bar{p}_i \cdot h_{c_i^x} \right) - K_0 \cdot \alpha_c \cdot (1 - \bar{\alpha}_v).
\end{aligned}$$

Substituting for  $\bar{p}_i = (\underline{p}_i + b_i)$

$$\begin{aligned}
& LB(TC_{Variables}) - UB(TC_{Variables}) \\
&= \sum_{c_i^x \in Variables} \left( \frac{K_{c_i^x}}{\underline{p}_i} + \underline{p}_i \cdot h_{c_i^x} - \frac{K_{c_i^x}}{(\underline{p}_i + b_i)} - (\underline{p}_i + b_i) \cdot h_{c_i^x} \right) + K_0 \cdot \alpha_c \cdot (\bar{\alpha}_v - \underline{\alpha}_v) \\
&= \sum_{c_i^x \in Variables} \left( \frac{b_i K_{c_i^x}}{\underline{p}_i (\underline{p}_i + b_i)} - b_i \cdot h_{c_i^x} \right) + K_0 \cdot \alpha_c \cdot (\bar{\alpha}_v - \underline{\alpha}_v). \tag{38}
\end{aligned}$$

Substituting for  $K_{c_i^x}$  and  $K_0$  using Eqs. (12) and (16) into Eq. (38) we get:

$$\begin{aligned}
& LB(TC_{Variables}) - UB(TC_{Variables}) \\
&= \sum_{c_i^x \in Variables} \left( -\frac{\alpha_c \bar{\alpha}_v b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right) + \alpha_c \cdot (\bar{\alpha}_v - \underline{\alpha}_v) \tag{39}
\end{aligned}$$

In order to simplify the expression we define for each  $c_i^x \in Variables$

$$\delta_i = \frac{\frac{\underline{p}_i - 1}{\underline{p}_i}}{\frac{\bar{p}_i - 1}{\bar{p}_i}}.$$

Substituting for  $\bar{p}_i = (\underline{p}_i + b_i)$

$$\begin{aligned}
\delta_i &= \frac{\frac{\underline{p}_i - 1}{\underline{p}_i}}{\frac{\underline{p}_i + b_i - 1}{\underline{p}_i + b_i}} = \frac{(\underline{p}_i - 1)(\underline{p}_i + b_i)}{\underline{p}_i (\underline{p}_i + b_i - 1)} = 1 - \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \\
&\Rightarrow \frac{\bar{p}_i - 1}{\bar{p}_i} \delta_i = \frac{\underline{p}_i - 1}{\underline{p}_i}.
\end{aligned}$$

Note that substituting  $\frac{\underline{p}_i - 1}{\underline{p}_i}$  into Eq. (7) using Eq. (6) we get that:

$$\bar{\alpha}_v \prod_{c_i^x \in Variables} \delta_i = \underline{\alpha}_v. \tag{40}$$

We substitute this expression into Eq. (39)

$$\begin{aligned}
& LB(TC_{Variables}) - UB(TC_{Variables}) \\
&= \alpha_c \sum_{c_i^x \in Variables} \left( -\frac{b_i \bar{\alpha}_v}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right) + \alpha_c \cdot \left( \bar{\alpha}_v - \bar{\alpha}_v \prod_{c_i^x \in Variables} \delta_i \right) \\
&= \alpha_c \left( \sum_{c_i^x \in Variables} \left( -\frac{b_i \bar{\alpha}_v}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right) + \bar{\alpha}_v \left( 1 - \prod_{c_i^x \in Variables} \delta_i \right) \right). \tag{41}
\end{aligned}$$

Note that

$$\begin{aligned}
& \prod_{c_i^x \in Variables} \delta_i = \prod_{c_i^x \in Variables} \left( 1 - \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right) \\
&= 1 - \sum_{c_i^x \in Variables} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} + \\
& \quad \sum_{\substack{c_i^x \in Variables \\ c_j^x \in Variables \\ i \neq j}} \left( \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \cdot \frac{b_j}{\underline{p}_j (\underline{p}_j + b_j - 1)} \right) - \dots + \prod_{c_i^x \in Variables} \left( \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right). \tag{42}
\end{aligned}$$

We denote the series in Eq. (42) as  $a_1, a_2, \dots, a_n$ , where  $a_1 = 1, a_2 = -\sum_{c_i^x \in Variables} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)}, \dots, a_n = \prod_{c_i^x \in Variables} \left( \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right)$ .  $a_3$  is a sum of  $n(n-1)$  elements where each element is a multiplication  $\frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \cdot \frac{b_j}{\underline{p}_j (\underline{p}_j + b_j - 1)}$  for each combination  $i \neq j, c_i^x, c_j^x \in Variables$ . Similarly,  $a_l$  is a sum of  $\binom{n}{l-1}$  elements for each combination of  $l-1$  commodities in  $Variables$  where each element is a multiple of  $(l-1)$  elements of the form  $\prod \left( \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right)$ . Each element in  $a_l$  is at least  $\frac{\underline{p}_1 (\underline{p}_1 + b_1 - 1)}{b_1}$  times bigger than any element in  $a_{l+1}$ ; however, there are  $(n-l)$  times more elements in  $a_{l+1}$  than in  $a_l$ . Since  $\frac{\underline{p}_1 (\underline{p}_1 + b_1 - 1)}{b_1 (n-l)} > 1$ , we have  $|a_l| > |a_{l+1}|$ . Therefore, we can upper bound the series in Eq. (42)

$$\begin{aligned}
& \prod_{c_i^x \in Variables} \delta_i \\
&< 1 - \sum_{c_i^x \in Variables} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} + \sum_{\substack{c_i^x \in Variables \\ c_j^x \in Variables \\ i \neq j}} \left( \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \cdot \frac{b_j}{\underline{p}_j (\underline{p}_j + b_j - 1)} \right). \tag{43}
\end{aligned}$$

Since  $\forall i : \underline{p}_1 \leq \underline{p}_i$  and  $\forall i : b \geq b_i > 1$ , we can upper bound the second summation in Eq. (43) by replacing

$\underline{p}_i$  and  $\underline{p}_j$  with  $\underline{p}_1$  and  $b_i, b_j$  in the numerator with  $b$ , and in the denominator with 1. Therefore

$$\begin{aligned}
& \prod_{c_i^x \in \text{Variables}} \delta_i \\
& < 1 - \sum_{c_i^x \in \text{Variables}} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} + \sum_{\substack{c_i^x \in \text{Variables} \\ c_j^x \in \text{Variables} \\ i \neq j}} \left( \frac{b}{\underline{p}_1 (\underline{p}_1 + 1 - 1)} \cdot \frac{b}{\underline{p}_1 (\underline{p}_1 + 1 - 1)} \right) \\
& < 1 - \sum_{c_i^x \in \text{Variables}} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} + b^2 n^2 \cdot \left( \frac{1}{\underline{p}_1^2} \right)^2 \\
& = 1 - \sum_{c_i^x \in \text{Variables}} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} + \frac{b^2 n^2}{\underline{p}_1^4}. \tag{44}
\end{aligned}$$

According to Condition 3,  $n < \underline{p}_1^{\frac{1}{6b}}$ , replacing  $n$  with the upper bound of  $\underline{p}_1^{\frac{1}{6b}}$  in Eq. (44),

$$\begin{aligned}
& \prod_{c_i^x \in \text{Variables}} \delta_i < 1 - \sum_{c_i^x \in \text{Variables}} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} + \frac{b^2 \underline{p}_1^{\frac{2}{6b}}}{\underline{p}_1^4} \\
& = 1 - \sum_{c_i^x \in \text{Variables}} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} + \frac{b^2}{\underline{p}_1^{4 - \frac{1}{3b}}}.
\end{aligned}$$

Replacing  $\prod_{c_i^x \in \text{Variables}} \delta_i$  into Eq. (41)

$$\begin{aligned}
& LB(TC_{\text{Variables}}) - UB(TC_{\text{Variables}}) \\
& > \alpha_c \left( \sum_{c_i^x \in \text{Variables}} \left( -\frac{b_i \bar{\alpha}_v}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right) + \bar{\alpha}_v \left( 1 - \left( 1 - \sum_{c_i^x \in \text{Variables}} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} + \frac{b^2}{\underline{p}_1^{4 - \frac{1}{3b}}} \right) \right) \right) \\
& = \alpha_c \bar{\alpha}_v \left( \sum_{c_i^x \in \text{Variables}} \left( -\frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right) + \left( \sum_{c_i^x \in \text{Variables}} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} - \frac{b^2}{\underline{p}_1^{4 - \frac{1}{3b}}} \right) \right) \\
& = -b^2 \alpha_c \bar{\alpha}_v \underline{p}_1^{\frac{1}{3b} - 4}.
\end{aligned}$$

■

**Claim 6**  $LB(TC_{\text{Clauses}}(S)) - TC_{\text{Clauses}}(S') > b^2 \alpha_c \bar{\alpha}_v \underline{p}_1^{\frac{1}{3b} - 4}$

**Proof.** Using Eqs. (35) and (36) with  $K_0 = 0$  (see Eq. (16)) we get:

$$\begin{aligned}
TC_{\text{Clauses}}(S') &= \sum_{c_r^\omega \in \text{Clauses}} \left( \frac{K_{c_r^\omega}}{t_{c_r^\omega}^*} + t_{c_r^\omega}^* \cdot h_{c_r^\omega} \right) \\
LB(TC_{\text{Clauses}}(S)) &= \sum_{c_r^\omega \in \text{Clauses}} \left( \frac{K_{c_r^\omega}}{t_{c_r^\omega}^*} + t_{c_r^\omega}^* \cdot h_{c_r^\omega} \right) + \\
&\quad \alpha_c \cdot \underline{\alpha}_v \cdot \left( 1 - \prod_{c_r^\omega \in F} \frac{t_{c_r^\omega}^* - 1}{t_{c_r^\omega}^*} \right).
\end{aligned}$$

Since  $F = \{c_r^\omega\}$  and as defined in Eq. (13),  $t_{c_r^\omega}^* = P(z_i) \cdot P(z_j) \cdot P(z_s)$ , we get:

$$\begin{aligned} & LB(TC_{Clauses}(S)) \\ &= \sum_{c_r^\omega \in Clauses} \left( \frac{K_{c_r^\omega}}{t_{c_r^\omega}^*} + t_{c_r^\omega}^* \cdot h_{c_r^\omega} \right) + \alpha_c \cdot \underline{\alpha}_v \frac{1}{t_{c_r^\omega}^*} \\ &= \sum_{c_r^\omega \in Clauses} \left( \frac{K_{c_r^\omega}}{t_{c_r^\omega}^*} + t_{c_r^\omega}^* \cdot h_{c_r^\omega} \right) + \alpha_c \cdot \underline{\alpha}_v \frac{1}{P(z_i) \cdot P(z_j) \cdot P(z_s)}. \end{aligned}$$

Therefore,

$$\begin{aligned} & TC_{Clauses}(S) - TC_{Clauses}(S') \\ &= \alpha_c \cdot \underline{\alpha}_v \frac{1}{P(z_i) \cdot P(z_j) \cdot P(z_s)}. \end{aligned}$$

Since  $\forall i : P(z_i) < \bar{p}_n < B\underline{p}_1$ , we can lower bound this expression by replacing  $P(z_i)$ ,  $P(z_j)$  and  $P(z_s)$  with  $B\underline{p}_1$ . Therefore

$$\begin{aligned} & TC_{Clauses}(S) - TC_{Clauses}(S') \\ &> \alpha_c \cdot \underline{\alpha}_v \frac{1}{B^3 \underline{p}_1^3}. \end{aligned}$$

Recall that according to Eq. (40),  $\bar{\alpha}_v \prod_{c_i^x \in Variables} \delta_i = \underline{\alpha}_v$  and therefore,

$$\begin{aligned} & TC_{Clauses}(S) - TC_{Clauses}(S') \\ &> \alpha_c \cdot \bar{\alpha}_v \prod_{c_i^x \in Variables} \delta_i \cdot \frac{1}{B^3 \underline{p}_1^3}. \end{aligned}$$

We can lower bound  $\prod_{c_i^x \in Variables} \delta_i$  using Eq. (42),

$$\begin{aligned} & TC_{Clauses}(S) - TC_{Clauses}(S') \\ &> \alpha_c \cdot \bar{\alpha}_v \left( 1 - \sum_{c_i^x \in Variables} \frac{b_i}{\underline{p}_i (\underline{p}_i + b_i - 1)} \right) \cdot \frac{1}{B^3 \underline{p}_1^3}. \end{aligned}$$

Since  $\forall i : \underline{p}_i \geq \underline{p}_1$  and  $\forall i : b_i < b$ , we can lower bound this expression by replacing  $\underline{p}_i$  with  $\underline{p}_1$  and  $b_i$  in the numerator with  $\tilde{b}$  and in the denominator with 1. Therefore:

$$\begin{aligned} & TC_{Clauses}(S) - TC_{Clauses}(S') \\ &> \alpha_c \cdot \bar{\alpha}_v \left( 1 - \frac{bn}{\underline{p}_1^2} \right) \cdot \frac{1}{B^3 \underline{p}_1^3}. \end{aligned}$$

Substituting for  $B$  according to Condition 2 we have:

$$\begin{aligned} & TC_{Clauses}(S) - TC_{Clauses}(S') \\ &> \frac{\alpha_c \cdot \bar{\alpha}_v}{\left( \tilde{b} \log n \right)^{3\tilde{b}} \underline{p}_1^3} \left( 1 - \frac{bn}{\underline{p}_1^2} \right) > \frac{\alpha_c \cdot \bar{\alpha}_v}{n^{3\tilde{b}} \underline{p}_1^3} \left( 1 - \frac{bn}{\underline{p}_1^2} \right), \end{aligned} \tag{45}$$

were the second inequality holds for any  $n > 3536$  even for the upper bound on  $\tilde{b}$  attained by the *Polymath8*

project [31]. According to Condition 3,  $n < \underline{p}_1^{\frac{1}{6b}}$ , replacing  $n$  with the upper bound  $\underline{p}_1^{\frac{1}{6b}}$  into Eq. (45) we get,

$$\begin{aligned}
& TC_{Clauses}(S) - TC_{Clauses}(S') \\
& > \frac{\alpha_c \cdot \bar{\alpha}_v}{\underline{p}_1^{3.5}} \left( 1 - \frac{b}{\underline{p}_1^{2 - \frac{1}{6b}}} \right) > \frac{\alpha_c \cdot \bar{\alpha}_v}{\underline{p}_1^{3.5}} \left( 1 - \frac{b}{\underline{p}_1^{2 - \frac{1}{6b}}} \right) \\
& > \frac{\alpha_c \cdot \bar{\alpha}_v}{\underline{p}_1^{3.5}} \left( \frac{b^2}{\underline{p}_1^{0.5 - \frac{1}{6b}}} \right) = \frac{b^2 \alpha_c \bar{\alpha}_v}{\underline{p}_1^{4 - \frac{1}{3b}}},
\end{aligned}$$

where the last inequality holds for  $\underline{p}_1 > 2362$  ( $n \geq 2$  according to Condition 3) even for the upper bound on  $\tilde{b}$  attained by the *Polymath8* project [31]. ■

Therefore,

$$\begin{aligned}
& LB(TC(S)) - UB(TC(S')) \\
& = TC_{Constants}(S) + LB(TC_{Variables}(S)) \\
& \quad + TC_{Clauses}(S) - TC_{Constants}(S') - UB(TC_{Variables}(S')) - TC_{Clauses}(S') \\
& = LB(TC_{Variables}(S)) - UB(TC_{Variables}(S')) + TC_{Clauses}(S) - TC_{Clauses}(S') \\
& > \frac{b^2 \alpha_c \bar{\alpha}_v}{\underline{p}_1^{4 - \frac{2}{3b}}} - \frac{b^2 \alpha_c \bar{\alpha}_v}{\underline{p}_1^{4 - \frac{2}{3b}}} = 0.
\end{aligned}$$

**Conclusion 1** *A solution to  $\Gamma$  that reflects an assignment  $\alpha$  that satisfies  $\varphi$  costs less than any solution to  $\Gamma$  that reflects an assignment  $\alpha'$  that does not satisfy  $\varphi$ . Therefore, solving PJRP is at least as hard as solving 3SAT.*

## 4 $\mathcal{NP}$ -Hardness of the Periodic Joint Replenishment Problem with finite horizon

The model of the finite time horizon is similar to the model of the infinite time horizon; however, since the time horizon is finite it is possible that for a commodity  $c$  the last cycle will not be a whole one. In the finite model we assume a time horizon of  $T$  periods. Similarly to the infinite time horizon, we analyzed the standalone *problem* cost function. In a case that  $\text{mod}(T, t_c) = 0$ , for a commodity  $c$ , the last cycle is a full one. In this case, the average periodic cost, denoted by  $\tilde{g}(t_c)$  is equal to  $g(t_c)$ . The expression for the average periodic cost as a function of the cycle time  $t_c$  in the case  $\text{mod}(T, t_c) \neq 0$  is a complex one. In order to avoid this situation for the same reduction defined in Section 3.1, we define

$$T = \prod_{i=2}^{(\bar{p}_n)^3} i,$$

This guarantees that all the cycle times that were analyzed in Section 3.2 are of the form  $t_c$  where  $\text{mod}(T, t_c) = 0$ . Therefore, the observations from Section 3.2 apply for the finite horizon model. However, since  $T$  is not polynomial in  $n$ , the problem is  $\mathcal{NP}$ -hard but not necessarily strongly  $\mathcal{NP}$ -hard.

## 5 Summary

In this paper we answer the long-standing open question regarding the computational complexity of *PJRP* with integer cycle times for a finite time horizon as well as for an infinite time horizon. We provided a proof that *PJRP* with integer cycle times and an infinite time horizon is strongly  $\mathcal{NP}$ -hard and that *PJRP* with integer cycle times and a finite time horizon is  $\mathcal{NP}$ -hard.

Another important problems yet to be answered is defining the computational complexity of *PJRP* with non-integer cycle times and of the strict *PJRP*.

**Acknowledgement 1** We thank Professor Retsef Levi for challenging us to solve the long-lasting question of the complexity of the PJRP.

**Acknowledgement 2** We thank and acknowledge the help of Noga Glassner in preparing this paper.

## References

- [1] Arkin, E., Joneja, D., & Roundy, R. (1989). Computational complexity of uncapacitated multi-echelon production planning problems. *Operations Research Letters*, 8(2), 61-66.
- [2] Fung, R. Y. K., & Ma, X. (2001). A new method for joint replenishment problems. *Journal of the Operational Research Society*, 52(3), 358-362.
- [3] Goyal, S. K. (1973). Determination of economic packaging frequency for items jointly replenished. *Management Science*, 20(2), 232-235.
- [4] Goyal, S. K. (1975). Analysis of joint replenishment inventory systems with resource restriction. *Operational Research Quarterly*, 197-203.
- [5] Goyal, S. K., & Deshmukh, S. G. (1993). Discussion A note on ‘The economic ordering quantity for jointly replenishing items’. *The International Journal of Production Research*, 31(12), 2959-2961.
- [6] Goyal, S. K., & Belton, A. S. (1979). On” A Simple Method of Determining Order Quantities in Joint Replenishments under Deterministic Demand. *Management Science*, 604-604.
- [7] Goyal, S. K., & Satir, A. T. (1989). Joint replenishment inventory control: deterministic and stochastic models. *European Journal of Operational Research*, 38(1), 2-13.
- [8] Guy, R. K. (1994). Gaps between Primes. Twin Primes.” §A8 in *Unsolved Problems in Number Theory*, 2nd edition New York: Springer-Verlag, 19-23, 1994.
- [9] Hardy, G. H. and Littlewood, J. E. (1923). Some Problems of ‘Partitio Numerorum.’ III. On the Expression of a Number as a Sum of Primes. *Acta Math.* 44, 1-70.
- [10] Hardy, G. H. and Wright, E. M. (1979) *An Introduction to the Theory of Numbers*, 5th ed. Oxford, England: Clarendon Press.
- [11] Jackson, P., Maxwell, W., & Muckstadt, J. (1985). The joint replenishment problem with a powers-of-two restriction. *IIE transactions*, 17(1), 25-32.
- [12] Joneja, D. (1990). The joint replenishment problem: new heuristics and worst case performance bounds. *Operations Research*, 38(4), 711-723.
- [13] Kaspi, M., & Rosenblatt, M. J. (1983). An improvement of Silver’s algorithm for the joint replenishment problem. *AIIE Transactions*, 15(3), 264-267.
- [14] Kaspi, M., & Rosenblatt, M. J. (1991). On the economic ordering quantity for jointly replenished items. *The International Journal of Production Research*, 29(1), 107-114.
- [15] Khouja, M., Michalewicz, Z., & Satoskar, S. S. (2000). A comparison between genetic algorithms and the RAND method for solving the joint replenishment problem. *Production Planning & Control*, 11(6), 556-564.
- [16] Khouja, M., & Goyal, S. (2008). A review of the joint replenishment problem literature: 1989–2005. *European Journal of Operational Research*, 186(1), 1-16.
- [17] Lee, F. C., & Yao, M. J. (2003). A global optimum search algorithm for the joint replenishment problem under power-of-two policy. *Computers & Operations Research*, 30(9), 1319-1333.



- [18] Levi, R., Roundy, R., Shmoys, D., & Sviridenko, M. (2008). A constant approximation algorithm for the one-warehouse multiretailer problem. *Management Science*, 54(4), 763-776.
- [19] Levi, R., Roundy, R. O., & Shmoys, D. B. (2006). Primal-dual algorithms for deterministic inventory problems. *Mathematics of Operations Research*, 31(2), 267-284.
- [20] Lu, L., & Posner, M. E. (1994). Approximation procedures for the one-warehouse multi-retailer system. *Management Science*, 40(10), 1305-1316.
- [21] Mertens, F. (1874). Ein Beitrag zur analytischen Zahlentheorie. *Journal für die reine und angewandte Mathematik*, 78, 46-62.
- [22] Moon, I. K., & Cha, B. C. (2006). The joint replenishment problem with resource restriction. *European Journal of Operational Research*, 173(1), 190-198.
- [23] Muckstadt, J. A., & Roundy, R. O. (1987). Multi-item, one-warehouse, multi-retailer distribution systems. *Management Science*, 33(12), 1613-1621.
- [24] Muckstadt, J. A., & Roundy, R. O. (1993). Analysis of multistage production systems. *Handbooks in Operations Research and Management Science*, 4, 59-131.
- [25] Nahmias, S. (2001). *Production and Operations Analysis*. McGraw-Hill. Irwin, New York.
- [26] Nocturne, D. J. (1973). Note-Economic Ordering Frequency for Several Items Jointly Replenished. *Management Science*, 19(9), 1093-1096.
- [27] Nonner, T., & Souza, A. (2009). A 5/3-approximation algorithm for joint replenishment with deadlines. In *Combinatorial Optimization and Applications* (pp. 24-35). Springer Berlin Heidelberg.
- [28] Nonner, T., & Sviridenko, M. (2013). An efficient polynomial-time approximation scheme for the joint replenishment problem. In *Integer Programming and Combinatorial Optimization* (pp. 314-323). Springer Berlin Heidelberg.
- [29] Olsen, A. L. (2005). An evolutionary algorithm to solve the joint replenishment problem using direct grouping. *Computers & Industrial Engineering*, 48(2), 223-235.
- [30] Papadimitriou, C., & Yannakakis, M. (1988, January). Optimization, approximation, and complexity classes. In *Proceedings of the twentieth annual ACM symposium on Theory of computing* (pp. 229-234). ACM.
- [31] Polymath8 project (2015). [http://michaelnielsen.org/polymath1/index.php?title=Bounded\\_gaps\\_between\\_primes](http://michaelnielsen.org/polymath1/index.php?title=Bounded_gaps_between_primes)
- [32] Porras, E. M., & Dekker, R. (2004). On the efficiency of optimal algorithms for the joint replenishment problem: a comparative study (No. EI 2004-33). *Econometric Institute Research Papers*.
- [33] Porras, E. M., & Dekker, R. (2005). Generalized Solutions for the joint replenishment problem with correction factor (No. EI 2005-19). *Econometric Institute Research Papers*.
- [34] Porras, E. M., & Dekker, R. (2006). An efficient optimal solution method for the joint replenishment problem with minimum order quantities. *European Journal of Operational Research*, 174(3), 1595-1615.
- [35] Ribenboim, P. (1996). *The new book of prime number records*. Springer.
- [36] Rosser, J. B., Schoenfeld, L. (1962). Approximate formulas for some functions of prime numbers. *Illinois J. Math*, 6, 64-94.
- [37] Roundy, R. (1985). 98%-effective integer-ratio lot-sizing for one-warehouse multi-retailer systems. *Management science*, 31(11), 1416-1430.
- [38] Schulz, A. S., & Telha, C. (2011). Approximation algorithms and hardness results for the joint replenishment problem with constant demands. In *Algorithms-ESA 2011* 628-639. Springer Berlin Heidelberg.

- [39] Segev, D. (2013). An approximate dynamic-programming approach to the joint replenishment problem. *Mathematics of Operations Research*, 39(2), 432-444.
- [40] Shanks, D. (1993) Solved and Unsolved Problems in Number Theory, 4th ed. New York: Chelsea, 30.
- [41] Shu, F. T. (1971). Economic ordering frequency for two items jointly replenished. *Management Science*, 17(6), B-406.
- [42] Silver, E. A. (1976). A simple method of determining order quantities in joint replenishments under deterministic demand. *Management Science*, 22(12), 1351-1361.
- [43] Teo, C. P., & Bertsimas, D. (2001). Multistage lot sizing problems via randomized rounding. *Operations Research*, 49(4), 599-608.
- [44] Van Eijs, M. J. G. (1993). A note on the joint replenishment problem under constant demand. *Journal of the Operational Research Society*, 185-191.
- [45] Viswanathan, S. (1996). A new optimal algorithm for the joint replenishment problem. *Journal of the Operational Research Society*, 936-944.
- [46] Viswanathan, S. (2002). On optimal algorithms for the joint replenishment problem. *Journal of the Operational Research Society*, 53(11), 1286-1290.
- [47] Wildeman, R. E., Frenk, J. B. G., & Dekker, R. (1997). An efficient optimal solution method for the joint replenishment problem. *European Journal of Operational Research*, 99(2), 433-444.
- [48] Zhang, Y., (2014). Bounded gaps between primes. *Annals of Mathematics* 179(3): 1121–1174.
- [49] Zipkin, P. H. (2000). Foundations of inventory management (Vol. 2). New York: McGraw-Hill.

## Appendix: Proof of Lemma 1.

**Proof.** In our proof we show that there is a set of at least  $nB$  pairs of consecutive primes that satisfy Conditions 1 and 2 within an interval  $[\underline{p}_1, B\underline{p}_1]$  for some  $B$  and  $\underline{p}_1$  that satisfy Condition 3. Then, we show that there is a subset of at least  $n$  pairs that satisfy Condition 4 as well. Finally, we show that this set could be identified in  $O\left(n^{6\tilde{b}+1} \log^{\tilde{b}} n\right)$  time.

According to [48] there are at least  $\frac{B\underline{p}_1}{\log^b B\underline{p}_1}$  2-tuple( $b$ ) prime pairs; hence, there are at least  $\frac{B\underline{p}_1}{\log^b B\underline{p}_1}$  pairs of consecutive primes with a gap of at most  $b$  between them. In order to lower bound the number of 2-tuple( $b$ ) primes in an interval  $[\underline{p}_1, B\underline{p}_1]$  we need an upper bound for the number of 2-tuple( $b$ ) primes smaller than  $\underline{p}_1$ . We use an extremely un-tight bound, assuming all primes smaller than  $\underline{p}_1$  are 2-tuple( $b$ ) primes. According to [36] there are at most  $\frac{1.255\underline{p}_1}{\log \underline{p}_1}$  prime numbers smaller than  $\underline{p}_1$ . Hence, there are no more than  $0.5 \frac{1.255\underline{p}_1}{\log \underline{p}_1}$  prime pairs smaller than  $\underline{p}_1$ . Therefore, the number of 2-tuple( $b$ ) prime pairs in an interval  $[\underline{p}_1, B \cdot \underline{p}_1]$ , denoted  $N_{VP}$ , satisfies:

$$N_{VP} \geq \frac{B\underline{p}_1}{\log^{\tilde{b}}(B\underline{p}_1)} - \frac{0.63\underline{p}_1}{\log \underline{p}_1}.$$

Note that

$$B\underline{p}_1 = \underline{p}_1^{\log_{\underline{p}_1} B} \cdot \underline{p}_1 = \underline{p}_1^{1+\log_{\underline{p}_1} B};$$

therefore,

$$\begin{aligned}
N_{VP} &\geq \frac{B\underline{p}_1}{\log^{\tilde{b}}\left(\frac{1+\log_{\underline{p}_1} B}{\underline{p}_1}\right)} - \frac{0.63\underline{p}_1}{\log \underline{p}_1} \\
&= \frac{B\underline{p}_1}{\left(1+\log_{\underline{p}_1} B\right)^{\tilde{b}} \log^{\tilde{b}} \underline{p}_1} - \frac{0.63\underline{p}_1}{\log \underline{p}_1} \\
&= \frac{\underline{p}_1}{\log \underline{p}_1} \left( \frac{B}{\left(1+\log_{\underline{p}_1} B\right)^{\tilde{b}} \log^{\tilde{b}-1} \underline{p}_1} - 0.63 \right). \tag{46}
\end{aligned}$$

We set

$$\begin{aligned}
B &= \log^{\tilde{b}} \underline{p}_1 \\
\implies B^{\frac{\tilde{b}-1}{\tilde{b}}} &= \log^{\tilde{b}-1} \underline{p}_1
\end{aligned} \tag{47}$$

Note that:

$$\left(1+\log_{\underline{p}_1} B\right) = \left(1+\tilde{b} \log_{\underline{p}_1} \log \underline{p}_1\right) < 1+\tilde{b}.$$

Thus, for a large enough  $x_s$  we have the numerator in Eq. (46) satisfy:

$$\left(1+\log_{\underline{p}_1} B\right)^{\tilde{b}} \log^{\tilde{b}-1} \underline{p}_1 < \left(1+\tilde{b}\right) B^{\frac{\tilde{b}-1}{\tilde{b}}} < B$$

and

$$\frac{B}{\left(1+\log_{\underline{p}_1} B\right)^{\tilde{b}} \log^{\tilde{b}-1} \underline{p}_1} > 1 \tag{48}$$

Substituting Eq. (48) into Eq. (46) we get:

$$N_{VP} > 0.37 \frac{\underline{p}_1}{\log \underline{p}_1}$$

Next, we need to find  $\underline{p}_1$  and show that it is not greater than  $n^{6\tilde{b}}$ . That is, we need to find  $\underline{p}_1$  such that  $\left[\underline{p}_1, B\underline{p}_1\right]$  contains at least  $2Bn$  pairs of 2-tuple( $b$ ) primes:

$$N_{VP} > 0.37 \frac{\underline{p}_1}{\log \underline{p}_1} > 2Bn \tag{49}$$

We substitute for  $B$  using Eq. (47) and, in order to satisfy Condition 3, we replace  $n$  with the upper bound of  $\underline{p}_1^{\frac{1}{6\tilde{b}}}$ ; hence the condition in Eq. (49) maintains that:

$$\begin{aligned}
0.37 \frac{\underline{p}_1}{\log \underline{p}_1} &> 2\underline{p}_1^{\frac{1}{6\tilde{b}}} \log^{\tilde{b}} \underline{p}_1 \\
\underline{p}_1^{1-\frac{1}{6\tilde{b}}} &> 5.4 \log^{\tilde{b}+1} \underline{p}_1
\end{aligned}$$

Let us extract the base 2 logarithm of both sides of the inequality:

$$\begin{aligned} \left(1 - \frac{1}{6\tilde{b}}\right) \log \underline{p}_1 &> (\tilde{b} + 1) (\log \log \underline{p}_1) + 2.435 \\ \frac{\log \underline{p}_1}{\log \log \underline{p}_1} &> \frac{\tilde{b} + 1 + \frac{2.435}{(\log \log \underline{p}_1)}}{1 - \frac{1}{6\tilde{b}}} \end{aligned}$$

Condition 3 guaranties that for  $n > 2$  we have  $\log \log \underline{p}_1 > 2.435$ ; hence,  $\frac{2.435}{(\log \log \underline{p}_1)} < 1$ . Let us look at the right side of the inequality:

$$\frac{\tilde{b} + 1 + \frac{2.435}{(\log \log \underline{p}_1)}}{1 - \frac{1}{6\tilde{b}}} < \frac{\tilde{b} + 2}{1 - \frac{1}{6\tilde{b}}} = \frac{6\tilde{b}^2 + 12\tilde{b}}{6\tilde{b} - 1} < \frac{(6\tilde{b} - 1)(\tilde{b} + 3)}{6\tilde{b} - 1} = \tilde{b} + 3$$

Thus, it is sufficient to find a  $\underline{p}_1$  that satisfies:

$$\begin{aligned} \frac{\log \underline{p}_1}{\log \log \underline{p}_1} &> \tilde{b} + 3. \\ \Rightarrow \underline{p}_1 &> \log^{\tilde{b}+3} \underline{p}_1 \end{aligned}$$

in order to satisfy Conditions 1-3. hence, for a sufficiently large  $n$  (and according to Conditions 3 a sufficiently large  $\underline{p}_1$ ) the condition in Eq. (49) is satisfied. That is, setting

$$\begin{aligned} \underline{p}_1 &> n^{6\tilde{b}} \\ \Rightarrow B = \log^{\tilde{b}} \underline{p}_1 &> (6\tilde{b} \log n)^{\tilde{b}} \end{aligned}$$

for a sufficiently large  $n$  guaranties that there are at least  $Bn$  pairs of 2-tuple( $b$ ) primes within  $[\underline{p}_1, B\underline{p}_1]$ ; hence, there are at least  $Bn$  pairs of consecutive primes with a gap of at most  $b$  between them.<sup>2</sup>

If we chose the set  $VP_2$  from within  $[\underline{p}_1, B\underline{p}_1]$  all pairs would satisfy Conditions 2 and 3.

Let us now greedily chose pairs of consecutive primes that satisfy Condition 1 starting from the smallest pair greater than  $\underline{p}_1$  and adding them to  $VP_2$  and  $VP$  as long as Condition 4 is satisfied with respect to the elements already chosen to be in  $VP$ .

Since  $\underline{p}_1 \leq B\underline{p}_1$ , each prime in  $VP$  can be a factor of at most  $B - 1$  numbers within  $[\underline{p}_1, B\underline{p}_1]$ . Since  $|VP| = 2n$  there are at most  $2n(B - 1)$  numbers that are factored by an element in  $VP$ ; thus there are at most  $2n(B - 1)$  pairs of consecutive primes that do not satisfy Condition 4 and at least  $n$  that do. Hence, the sets  $VP_2$  and  $VP$  that satisfy Conditions 1-4 are found in a range that is polynomial to  $n$ .

Next we show that the sets  $VP_2$  and  $VP$  could be found in polynomial time by showing that all primes within the range  $[0, B\underline{p}_1]$  could be found in polynomial time.

Using Sieve of Eratosthenes [35] the first  $k$  prime numbers could be found in  $O(k^2)$  time. Since there is an upper bound of no more than  $1.25506 \frac{B\underline{p}_1}{\log B\underline{p}_1}$  prime in the range  $[0, B\underline{p}_1]$  (see [36]), finding the prime

---

<sup>2</sup>The size of  $n$  required to satisfy this constraint for the values of  $b$  and  $\tilde{b}$  found by [31] is extremely large (greater than  $2^{470}$ ). However, for any smaller  $n$  empirical evidence show that the twin prime conjecture [8] and the first Hardy-Littlewood conjecture [9] hold (Hardy and Wright [10] note that "the evidence, when examined in detail, appears to justify the conjecture," and Shanks [40] stated "the evidence is overwhelming"). Using twin primes the  $n$  required drops dramatically. Thus, our reduction is permissible for any  $n \geq 12$  (the whole analysis and bounds become tighter).

numbers in the range  $\left[0, B_{\underline{p}_1}\right]$  can be done in

$$O\left(\left(\frac{B_{\underline{p}_1}}{\log B_{\underline{p}_1}}\right)^2\right),$$

which is polynomial. Once the primes are identified, the greedy method to construct  $VP_2$  and  $VP$  requires  $O\left(nB_{\underline{p}_1}\right) = O\left(n^{6\tilde{b}+1} \log^{\tilde{b}} n\right)$  time. ■

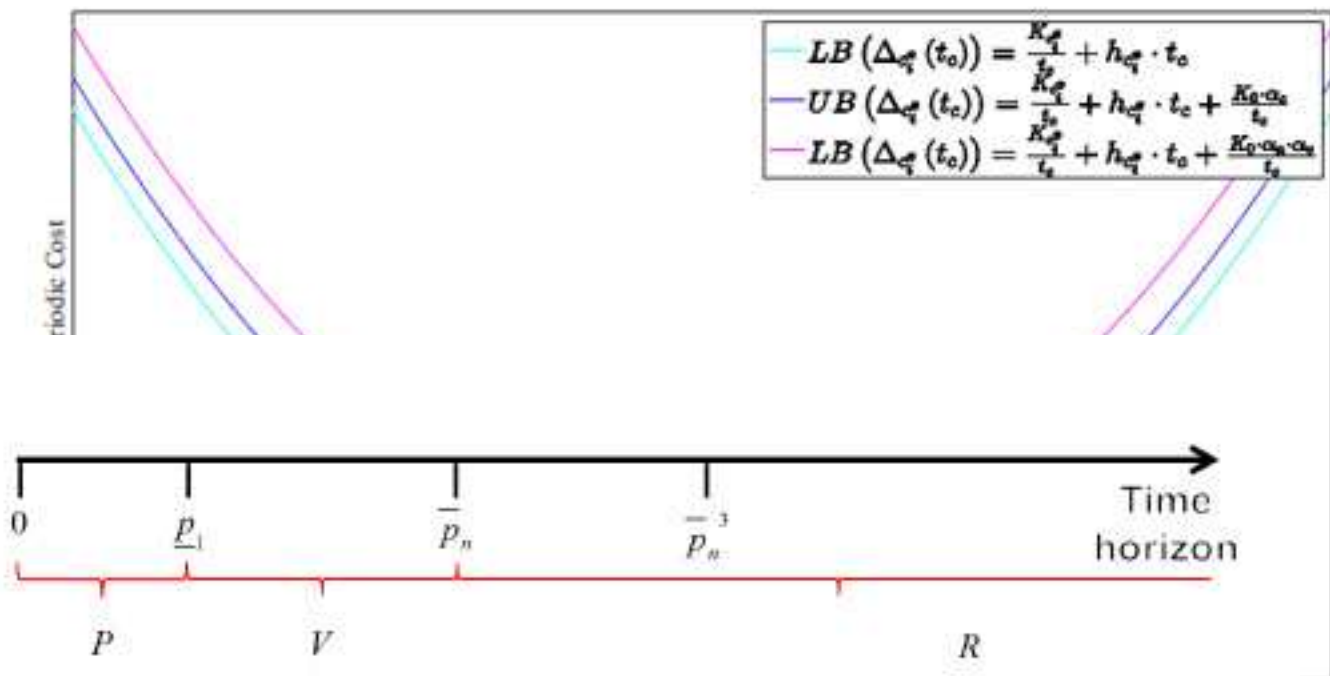


Figure 1: Time horizon segmentation